

Quadratic Equations & Inequations

INTRODUCTION TO POLYNOMIALS

Section - 1

1.1 Real Polynomial

Let $a_0, a_1, a_2, \dots, a_n$ be real numbers and x is a real variable. Then,

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

is called a real polynomial of real variable x with real coefficient.

For example :

$f(x) = 2x^2 + 3x + 1, f(x) = x + 3, f(x) = 5x^4 + 3x^2 - 4x - 1$ are some examples of real polynomials.

Note : How to identify a polynomial ?

Polynomial in x should be an expression in terms of various powers of x where every power should be a positive integer.

For example :

$$f(x) = x + \frac{1}{x} + 2, f(x) = x^2 + x^{1/2} + 3,$$

$$f(x) = x^{-2} - x^{-1} + 1 \text{ are not polynomials.}$$

1.2 Degree of a Polynomial

The degree of a real polynomial is the highest power of x in the polynomial.

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0x^0$$

is a polynomial of degree n , as highest power of x is n where n is a positive integer and $a_n \neq 0$.

For example :

$$f(x) = x^3 - 2x^2 + x - 1 \text{ is a polynomial of degree 3.}$$

$$f(x) = x^5 + x^2 - 1 \text{ is a polynomial of degree 5.}$$

➤ **Linear Polynomial :** Polynomial of degree one is known as linear polynomial.

For example :

$$f(x) = 2x + 3, f(x) = -x + 5 \text{ are linear polynomials.}$$

Note : $f(x) = ax + b$ is a general degree one polynomial known as linear polynomial.

$$a \neq 0.$$

➤ **Quadratic Polynomial :** Polynomial of degree two is known as quadratic polynomial.

For example :

$$f(x) = x^2 + x + 1, f(x) = -x^2 + 2x - 1 \text{ are quadratic polynomials.}$$

Note : $f(x) = ax^2 + bx + c, a \neq 0$ is a general degree two polynomial known as quadratic polynomial.

1.3 Polynomial Equation

If $y = f(x)$ is a real polynomial of degree n , then $f(x) = 0$ is the corresponding real polynomial equation of degree n .

For example :

If $f(x) = x^2 - 2x - 8$ is a quadratic polynomial, then $x^2 - 2x - 8 = 0$ is the corresponding quadratic equation.

1.4 Roots of an Equation

Roots of an equation in x are those values of x which satisfy the equation.

OR

If $f(\alpha) = 0$, then $x = \alpha$ is the root of the equation $f(x) = 0$.

For example :

$x = 2, x = 3$ are roots of $x^2 - 5x + 6 = 0$ because when we replace $x = 2$ or $x = 3$ in the equation,

we get : $0 = 0$. This implies $x = 2$ and $x = 3$ satisfy equation. Hence $x = 2$ and $x = 3$ are roots of the equation.

Note : ➤ Real roots of an equation $f(x) = 0$ are the x - co-ordinates of the points where graph of $y = f(x)$ intersects X -axis.
 ➤ An equation of degree n has n roots. (**not necessarily all real**) .

QUADRATIC EQUATION & INEQUATION

Section - 2

2.1 Introduction

The standard form of the quadratic equation is :

$$ax^2 + bx + c = 0 \quad \text{where } a, b, c \text{ are real numbers and } a \neq 0.$$

2.2 Roots of a Quadratic Equation

Roots of a quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0, a, b, c \in R$) are given by :

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Sum of the roots $= \alpha + \beta = -\frac{b}{a}$
- Product of roots $= \alpha \beta = \frac{c}{a}$
- Factorized form of $ax^2 + bx + c = a(x - \alpha)(x - \beta)$.
- If S be the sum and P be the product of roots, then quadratic equation is : $x^2 - Sx + P = 0$.

Illustrating the Concept :

(a) If α and β are the roots of equation $ax^2 + bx + c = 0$, find the value of following expressions.

(i) $\alpha^2 + \beta^2$ (ii) $\alpha^3 + \beta^3$ (iii) $\alpha^4 + \beta^4$ (iv) $(\alpha - \beta)^2$ (v) $\alpha^4 - \beta^4$

SOLUTION :

In such type of problems, try to represent the given expression in terms of $\alpha + \beta$ (sum of roots) and $\alpha\beta$ (product of roots).

In the given problem :

$$\alpha + \beta = \frac{-b}{a}, \quad \alpha\beta = \frac{c}{a}$$

$$(i) \quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(-\frac{b}{a}\right)^2 - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

$$(ii) \quad \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = \left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right) = \frac{-b^3 + 3abc}{a^3}$$

$$(iii) \quad \alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2 \\ = \left(\frac{b^2 - 2ac}{a^2}\right)^2 - 2\frac{c^2}{a^2} = \frac{(b^2 - 2ac)^2 - 2c^2a^2}{a^4}$$

$$(iv) \quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \frac{b^2}{a^2} - \frac{4c}{a} = \frac{b^2 - 4ac}{a^2}$$

$$(v) \quad \alpha^4 - \beta^4 = (\alpha^2 + \beta^2)(\alpha + \beta)(\alpha - \beta)$$

$$= \left(\frac{b^2 - 2ac}{a^2}\right)\left(-\frac{b}{a}\right)\left(\pm\sqrt{\frac{b^2 - 4ac}{a^2}}\right) \quad \left[\text{Using (i) and } \left(\alpha - \beta = \pm \frac{\sqrt{D}}{a}\right)\right] \\ = \pm \frac{b}{a^4}(b^2 - 2ac)\sqrt{b^2 - 4ac}$$

(b) If α and β are the roots of equation $ax^2 + bx + c = 0$, form an equation whose roots are :

$$(i) \quad \alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha} \quad (ii) \quad \frac{1}{\alpha + \beta}, \frac{1}{\alpha} + \frac{1}{\beta}$$

SOLUTION :

We know that to form an equation whose roots are known we have to find sum and product of the roots.

$$(i) \quad \text{Sum (S)} = \left(\alpha + \frac{1}{\beta}\right) + \left(\beta + \frac{1}{\alpha}\right) = (\alpha + \beta) + \frac{(\alpha + \beta)}{\alpha\beta} = \frac{-b(a + c)}{ac}$$

$$\text{Product (P)} = \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right) = \alpha\beta + \frac{1}{\alpha\beta} + 2 = \frac{(c + a)^2}{ac}$$

The equation is : $x^2 - Sx + P = 0$

$$\Rightarrow x^2 - \left(\frac{-b(a + c)}{ac}\right)x + \frac{(c + a)^2}{ac} = 0$$

$$\Rightarrow acx^2 + b(c + a)x + (c + a)^2 = 0 \text{ is the required equation.}$$

$$(ii) \quad \text{Sum (S)} = \left(\frac{1}{\alpha + \beta}\right) + \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \left(\frac{1}{\alpha + \beta}\right) + \frac{(\alpha + \beta)}{\alpha\beta} = -\frac{(ac + b^2)}{bc}$$

$$\text{Product (P)} = \left(\frac{1}{\alpha + \beta}\right)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \left(\frac{1}{\alpha\beta}\right) = \frac{a}{c}$$

The equation is : $x^2 - Sx + P = 0$

$$\Rightarrow x^2 - \left(-\frac{(ac + b^2)}{bc}\right)x + \frac{a}{c} = 0 \quad \Rightarrow bcx^2 + (ac + b^2)x + ab = 0 \text{ is the required equation.}$$

2.3 Nature of Roots of a Quadratic Equation

Nature of roots of a quadratic equation $ax^2 + bx + c = 0$ means whether the roots are real or complex. By analyzing the expression $b^2 - 4ac$ (called as discriminant, D), one can get an idea about the nature of the roots as follows :

1. (a) If $D < 0$ ($b^2 - 4ac < 0$), then the roots of the quadratic equation are non – real i.e. complex roots.
 (b) If $D = 0$ ($b^2 - 4ac = 0$), then the roots are real and equal.
 Equal root = $-\frac{b}{2a}$
 (c) If $D > 0$ ($b^2 - 4ac > 0$), then the roots are real and unequal.
2. If D i.e. ($b^2 - 4ac$) is a perfect square and a, b and c are rational, then the roots are rational.
3. If D i.e. ($b^2 - 4ac$) is not a perfect square and a, b and c are rational, then roots are of the form $m + \sqrt{n}$ and $m - \sqrt{n}$.
4. If $a = 1, b, c \in \mathbb{I}$ and the roots are rational numbers, then the roots must be integer.
5. If a quadratic equation in x has more than two roots, then it is an identity in x (i.e. true for all real values of x) and $a = b = c = 0$.

Illustrating the Concept :

Comment upon the nature of roots of the following equations :

- (i) $x^2 + (a + b)x - c^2 = 0$
- (ii) $(a + b + c)x^2 - 2(a + b)x + (a + b - c) = 0$
- (iii) $(b - c)x^2 + (c - a)x + (a - b) = 0$
- (iv) $x^2 + 2(3a + 5)x + 2(9a^2 + 25) = 0$
- (v) $(y - a)(y - b) + (y - b)(y - c) + (y - c)(y - a) = 0$

SOLUTION :

To comment upon the nature of roots of quadratic equation we have to find 'D' (Discriminant)

- (i) Find Discriminant (D).
 $D = (a + b)^2 - 4(1)(-c^2) = (a + b)^2 + 4c^2$
 $\Rightarrow D \geq 0$, hence the roots are real.
- (ii) $D = 4(a + b)^2 - 4(a + b + c)(a + b - c)$
 $= 4[(a + b)^2 - (a + b)^2 + c^2]$
 $= 4c^2 = (2c)^2$
 $\Rightarrow D \geq 0$ and also a perfect square, hence the roots are rational.
- (iii) $D = (c - a)^2 - 4(b - c)(a - b)$
 $= c^2 + a^2 - 2ac - 4ab + 4b^2 + 4ac - 4bc$
 $= c^2 + a^2 + (2b)^2 - 4ab - 4bc + 2ac$
 $= (c + a - 2b)^2$
 $\Rightarrow D \geq 0$ and also a perfect square, hence the roots are rational.
- (iv) $D = 4(3a + 5)^2 - 8(9a^2 + 25) = -4(3a - 5)^2$
 $\Rightarrow D \leq 0$, so the roots are non real if $a \neq 5/3$ and real and equal if $a = 5/3$.

- (v) Simplifying the given equation ;

$$3y^2 - 2(a + b + c)y + (ab + bc + ca) = 0$$

Now $D = 4(a + b + c)^2 - 12(ab + bc + ca)$

$$= 4(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$= 2[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

Using : $(a^2 + b^2 + c^2 - ab - bc - ca) = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$

$$\Rightarrow D \geq 0, \text{ so the roots are real.}$$

Note : If $D = 0$, then $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$$\Rightarrow a = b = c \Rightarrow \text{if } a = b = c, \text{ then the roots are equal.}$$

2.4 Condition for Common Root(s)

Consider two quadratic equations :

$$ax^2 + bx + c = 0 \quad \text{and} \quad a'x^2 + b'x + c' = 0$$

- (a)
- For two common roots :**

In such a case, two equations should be identical. For that, the ratio of coefficients of x^2 , x and x^0 must be same,

i.e. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

- (b)
- For one common root :**

Let α be the common root of two equations. So α should satisfy the two equations.

$$\Rightarrow a\alpha^2 + b\alpha + c = 0 \quad \text{and} \quad a'\alpha^2 + b'\alpha + c' = 0$$

Solving the two equations by using **Cramer's Rule** (or cross multiplication method) :

$$\Rightarrow \frac{\alpha^2}{bc' - b'c} = \frac{-\alpha}{ac' - a'c} = \frac{1}{ab' - a'b}$$

$$\Rightarrow \alpha = \frac{a'c - ac'}{ab' - a'b}, \alpha^2 = \frac{bc' - b'c}{ab' - a'b}$$

$$\Rightarrow (bc' - b'c)(ab' - a'b) = (a'c - ac')^2$$

This is the condition for one root of two quadratic equations to be common.

Note : To find the common root between the two equations, make the coefficient of α^2 common and then subtract the two equations.

Illustration - 1

The equation whose roots are squares of the sum and the difference of the roots of the equation

$$2x^2 + 2(m + n)x + m^2 + n^2 = 0 \text{ is :}$$

(A) $x^2 - 4mnx + (m^2 - n^2)^2 = 0$

(B) $x^2 + 4mnx + (m^2 - n^2)^2 = 0$

(C) $x^2 - 4mnx - (m^2 - n^2)^2 = 0$

(D) $x^2 - 4mnx - (m^2 - n^2)^2 = 0$

SOLUTION : (C)

Let α, β be the roots of given equation.

$$\Rightarrow \alpha + \beta = -(m+n) \text{ and } \alpha\beta = \frac{(m^2 + n^2)}{2}$$

Now we have to make an equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$.

$$\text{Sum (S)} = (\alpha + \beta)^2 + (\alpha - \beta)^2 = 2(\alpha^2 + \beta^2) = 2[(\alpha + \beta)^2 - 2\alpha\beta] = 4mn$$

$$\text{Product (P)} = (\alpha + \beta)^2 \cdot (\alpha - \beta)^2 = (\alpha + \beta)^2 \cdot [(\alpha + \beta)^2 - 4\alpha\beta]$$

$$P = (m+n)^2 [(m+n)^2 - 2(m^2 + n^2)] = -(m^2 - n^2)^2$$

The equation is : $x^2 - Sx + P = 0$

$$\Rightarrow \text{The required equation is } x^2 - 4mnx - (m^2 - n^2)^2 = 0$$

Illustration - 2

The value of k , so that the equations $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ may have one root in common is :

- (A) $-3, -1$ (B) $-3, \frac{-27}{4}$ (C) $-1, -2$ (D) $3, \frac{27}{4}$

SOLUTION : (B)

Let α be the common root of two equations.

$$\text{Hence } 2\alpha^2 + k\alpha - 5 = 0 \text{ and } \alpha^2 - 3\alpha - 4 = 0$$

Solving the two equations ;

$$\frac{\alpha^2}{-4k - 15} = \frac{-\alpha}{-8 + 5} = \frac{1}{-6 - k} \quad [\text{Using : 2.4 (b)}]$$

$$\Rightarrow (-3)^2 = (4k + 15)(6 + k) \Rightarrow 4k^2 + 39k + 81 = 0$$

$$\Rightarrow k = -3 \text{ or } k = -27/4$$

Illustration - 3

If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a root in common, the relation between a, b and c is :

- (A) $a = 0$ or $a^3 + b^3 + c^3 = 3abc$ (B) $a = 0$ or $a^3 + b^3 + c^3 = -3abc$
 (C) $a = 0$ or $a^3 - b^3 - c^3 = 3abc$ (D) $a = 0$ or $a^3 + b^3 - c^3 = -3abc$

SOLUTION : (A)

Solve the two equations as done in last illustration :

$$ax^2 + bx + c = 0 \text{ and } bx^2 + cx + a = 0$$

$$\frac{x^2}{ba - c^2} = \frac{-x}{a^2 - bc} = \frac{1}{ac - b^2} \quad [\text{Using : 2.4 (b)}]$$

$$\Rightarrow (a^2 - bc)^2 = (ba - c^2)(ac - b^2)$$

$$\text{Simplify to get : } a(a^3 + b^3 + c^3 - 3abc) = 0$$

$$\Rightarrow a = 0 \text{ or } a^3 + b^3 + c^3 = 3abc$$

This is the relation between a, b and c .

Illustration - 4 If the equations $x^2 - ax + b = 0$ and $x^2 - cx + d = 0$ have one root in common and second equation has equal roots, then the value of ac is :

- (A) b (B) $2(b + d)$ (C) $b - d$ (D) bd

SOLUTION : (B)

The equation $x^2 - cx + d = 0$ has equal roots.

$$\Rightarrow D = 0 \Rightarrow D = c^2 - 4d = 0 \quad \dots\dots(i)$$

$$\Rightarrow x = \frac{c}{2} \text{ is the equal root of this equation.} \quad \left[\text{As equal roots of } ax^2 + bx + c = 0 \text{ are } x = \frac{-b}{2a} \right]$$

Now this should be the common root.

$$\therefore x = \frac{c}{2} \text{ will satisfy the first equation}$$

$$\Rightarrow \frac{c^2}{4} - a\left(\frac{c}{2}\right) + b = 0$$

$$\Rightarrow c^2 + 4b = 2ac \Rightarrow 4d + 4b = 2ac \quad [\text{Using (i)}]$$

$$\Rightarrow 2(d + b) = ac \quad \text{Hence } ac = 2(b + d)$$

Illustration - 5 If the ratio of roots of the equation $x^2 + px + q = 0$ be equal to the ratio of roots of the equation $x^2 + bx + c = 0$, then the value of $p^2c - b^2q$ =

- (A) -1 (B) 1 (C) 0 (D) None of these

SOLUTION : (C)

Let α, β be the roots of $x^2 + px + q = 0$ so, $\alpha + \beta = -p, \alpha\beta = q$

and also let γ, δ roots of $x^2 + bx + c = 0$ so, $\gamma + \delta = -b, \gamma\delta = c$

$$\text{Now, } \frac{\alpha}{\beta} = \frac{\gamma}{\delta} \Rightarrow \frac{(\alpha + \beta)^2}{(\alpha - \beta)^2} = \frac{(\gamma + \delta)^2}{(\gamma - \delta)^2} \quad (\text{Apply componendo-dividendo and take square on both sides})$$

$$\Rightarrow \frac{(\alpha + \beta)^2}{(\alpha + \beta)^2 - (\alpha - \beta)^2} = \frac{(\gamma + \delta)^2}{(\gamma + \delta)^2 - (\gamma - \delta)^2}$$

$$\Rightarrow \frac{(\alpha + \beta)^2}{4\alpha\beta} = \frac{(\gamma + \delta)^2}{4\gamma\delta} \Rightarrow \frac{p^2}{4q} = \frac{b^2}{4c} \Rightarrow p^2c = b^2q.$$

Illustration - 6 The condition that the equation $\frac{1}{x} + \frac{1}{x+b} = \frac{1}{m} + \frac{1}{m+b}$ has real roots that are equal in magnitude but opposite in sign is

- (A) $b^2 = m^2$ (B) $b^2 = 2m^2$ (C) $2b^2 = m^2$ (D) None of these

SOLUTION : (B)

Clearly $x = m$ is a root of the equation. Therefore, the other root must be $-m$. That is,

$$\begin{aligned} \frac{1}{-m} + \frac{1}{-m+b} &= \frac{1}{m} + \frac{1}{m+b} \Rightarrow \frac{1}{b-m} - \frac{1}{b+m} = \frac{2}{m} \\ \Rightarrow \frac{b+m-b+m}{b^2-m^2} &= \frac{2}{m} \Rightarrow 2m^2 = 2b^2 - 2m^2 \text{ or } 2m^2 = b^2. \end{aligned}$$

INEQUATIONS & INEQUALITIES

Section - 3

3.1 Set Theory

Introduction to Sets

[I] A set is a well-defined collection of objects or elements. Each element in a set is unique. Usually but not necessarily, a set is denoted by a capital letter *i.e.*, $A, B, \dots, U, V$ etc. and the elements are enclosed between curly brackets $\{ \}$. The elements are denoted by small letters $a, b, \dots, x, y$ etc. For example:

A = Set of all small English alphabets

$$= \{a, b, c, \dots, x, y, z\}$$

B = Set of all positive integers less than or equal to 10

$$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

R = Set of real numbers

$$= \{x : -\infty < x < \infty, x \in R\}$$

The elements of a set can be discrete (*e.g.* set of all English alphabets) or continuous (*e.g.* set of real numbers). The set may contain finite or infinite number of elements. A set may contain no element and such a set is called **Void set or Null set or empty set** and is denoted by ϕ (*phi*) or $\{ \}$. The number of elements of a set A is denoted by $n(A)$ and sometimes referred as **cardinality** of set A . Hence $n(\phi) = 0$ as it contains no element.

For example: The collection of girl students in a boys college is the empty set.

[II] A set is called a finite set if the process of counting of its different elements comes to an end and vice versa.

For example: The set of all divisors of a given natural number is a finite set.

Union of Sets

Union of two or more sets is the set of all elements that belong to any of these sets. The symbol used for union of sets is ' $\cup$ '.

i.e., $A \cup B$ = Union of set A and set $B = \{x : x \in A \text{ or } x \in B\}$

For example : If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 5, 6\}$ and $C = \{1, 2, 6, 8\}$, then $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 8\}$

Intersection of Sets

It is the set of all the elements, which are common to all the sets under consideration. The symbol used for intersection of sets is ' $\cap$ ' *i.e.* $A \cap B = \{x : x \in A \text{ and } x \in B\}$

For example : If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 6\}$ and $C = \{1, 2, 6, 8\}$, then $A \cap B \cap C = \{2\}$.

Remember that, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ where $n(A \cup B)$ means number of elements in A union B

Difference of Two Sets

The difference of set A to B denoted as $A - B$, is the set of those elements that are in the set A but not in the set B *i.e.* $A - B = \{x : x \in A \text{ and } x \notin B\}$

Similarly $B - A = \{x : x \in B \text{ and } x \notin A\}$. In general, $A - B \neq B - A$

For example : If $A = \{a, b, c, d\}$ and $B = \{b, c, e, f\}$, then $A - B = \{a, d\}$ and $B - A = \{e, f\}$

If $A - B = B - A \Rightarrow$ then $A = B$

3.2 Inequalities

The following are some very useful points to remember :

- $a \leq b \Rightarrow$ Either $a < b$ or $a = b$
- $a < b$ and $b < c \Rightarrow a < c$
- $a < b \Rightarrow a + c < b + c \forall c \in R$
- $a < b \Rightarrow -a > -b$ i.e. inequality sign reverses if both sides are multiplied by a negative number.
- $a < b$ and $c < d \Rightarrow a + c < b + d \Rightarrow a - d < b - c$.
- $a < b \Rightarrow ma < mb$ if $m > 0$ and $ma > mb$ if $m < 0$.
- $0 < a < b \Rightarrow a^r < b^r$ if $r > 0$ and $a^r > b^r$ if $r < 0$
- $\left(a + \frac{1}{a}\right) \geq 2 \forall a > 0$ and equality holds for $a = 1$.
- $\left(a + \frac{1}{a}\right) \leq -2 \forall a < 0$ and equality holds for $a = -1$.

3.3 Interval

An infinite continuous subset of R is called an interval.

3.4 Closed Interval

The set of real numbers between a and b (where $a < b$) also including the end points a and b is called a closed interval and is denoted by $[a, b]$. Thus $[a, b] = \{x \in R : a \leq x \leq b\}$

3.5 Open Interval

The set of real numbers between a and b (where $a < b$) but excluding the end points a and b is called an open interval and is denoted by (a, b) . Thus $(a, b) = \{x : a < x < b\}$.

The set of real numbers x such that $a < x \leq b$ is called a semi - open or semi -closed interval and is denoted by $(a, b]$. Similarly we have $[a, b) = \{x : a \leq x < b\}$.

The number $b - a$ is called length of the interval (a, b) or of $[a, b]$. The smallest and greatest elements in an open interval (a, b) do not exist.

3.6 Infinite Intervals

The set of all real numbers greater than a certain real number, say ' a ', is an infinite interval and is denoted by (a, ∞) . Thus $(a, \infty) = \{x : x > a\}$, and $[a, \infty) = \{x : x \geq a\}$

Similarly, we define $(-\infty, a) = \{x : x < a\}$ and $(-\infty, a] = \{x : x \leq a\}$

The infinite intervals have infinite length. In writing them, we use the symbol ∞ as notation only. The sides of $-\infty$ and ∞ in writing infinite intervals **must be kept open** because a real number is never equal to $-\infty$ or ∞ .

Note : In any interval the smaller value is to be written first. For example, suppose we want to write set of real numbers between 1 and 3 then we will denote it by (1, 3) and not by (3, 1).

QUADRATIC POLYNOMIAL

Section - 4

4.1 Introduction

The quadratic polynomial in x is $ax^2 + bx + c$; where a, b, c are real numbers and $a \neq 0$.

$ax^2 + bx + c$ is also known as the quadratic expression in x . Evidently $ax^2 + bx + c$ is a function in x . For different real values of x , we get different real values of $ax^2 + bx + c$.

So, in general quadratic expression is represented as : $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$

4.2 Graph of a Quadratic Polynomial

$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$

To draw the graph of $f(x)$, proceed according to following steps :

1. The shape of the curve $y = f(x)$ is **parabolic**.

2. For $a > 0$, the parabola opens upwards.

For $a < 0$, the parabola opens downwards.

3. **Intersection with axes :**

(i) **with X-axis**

➤ **For $D > 0$**

Parabola cuts X-axis in two points.

The points of intersection are $\alpha, \beta = \frac{-b \pm \sqrt{D}}{2a}$.

➤ **For $D = 0$**

Parabola touches X-axis in one point.

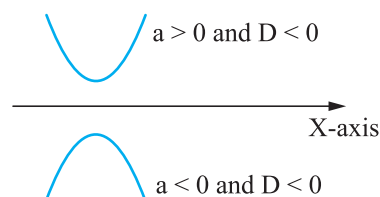
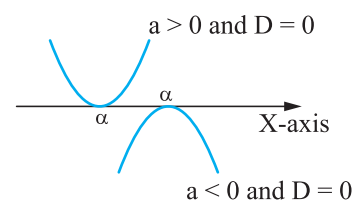
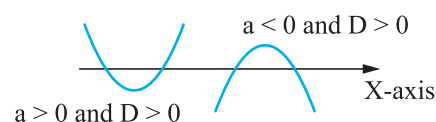
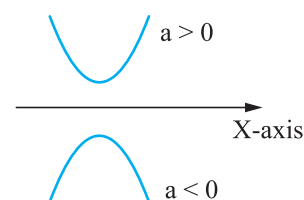
The point of contact is $\alpha = \frac{-b}{2a}$.

➤ **For $D < 0$**

Parabola does not cut X-axis at all i.e. no point of intersection with X-axis.

(ii) **with Y-axis**

The point of intersection with Y-axis is $(0, c)$ {put $x = 0$ in the quadratic polynomial}



4. Maximum and Minimum value of $f(x)$:

V is called as vertex of parabola.

The coordinates of $V \equiv \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

The line passing through vertex and parallel to the Y -axis is called as axis of symmetry.

The parabolic graph of a quadratic polynomial is symmetrical about axis of symmetry.

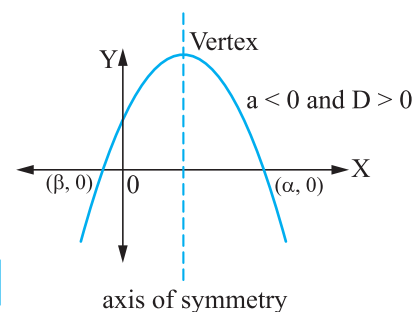
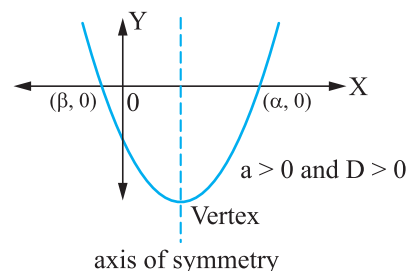
- $f(x)$ has minimum value at vertex if $a > 0$ and

$$f_{\min} = -\frac{D}{4a} \text{ at } x = -\frac{b}{2a}.$$

- $f(x)$ has maximum value at vertex if $a < 0$ and

$$f_{\max} = -\frac{D}{4a} \text{ at } x = -\frac{b}{2a}.$$

Note : Graph of any quadratic polynomial can be plotted by following steps (1) to (4).

**4.3 Sign of a Quadratic Polynomial**

Let $f(x) = ax^2 + bx + c$ where $a, b, c \in R$ and $a \neq 0$.

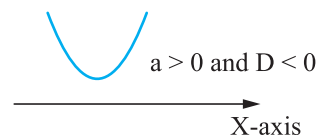
1. $a > 0, D < 0$:

As $a > 0$, parabola opens upward.

As $D < 0$, parabola does not intersect X -axis.

So $f(x) > 0$ for all $x \in R$.

i.e., $f(x)$ is positive for all value of x .

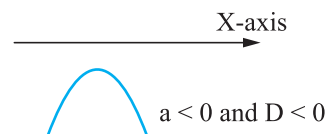
**2. $a < 0, D < 0$:**

As $a < 0$, parabola opens downward.

As $D < 0$, parabola does not intersect X -axis.

So $f(x) < 0$ for all $x \in R$.

i.e., $f(x)$ is negative for all value of x .

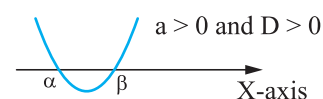
**3. $a > 0, D > 0$:**

As $a > 0$, parabola opens upward.

As $D > 0$, parabola intersects X -axis in two points say α, β ($\alpha < \beta$).

So $f(x) \geq 0$ for all $x \in (-\infty, \alpha] \cup [\beta, \infty)$ and $f(x) < 0$ for all $x \in (\alpha, \beta)$.

i.e., $f(x)$ is positive for some values of x and negative for other values for x .



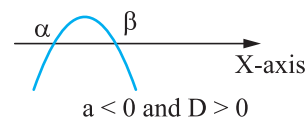
4. $a < 0, D > 0$:

As $a < 0$, parabola opens downward.

As $D > 0$, parabola intersect X-axis in two points say α, β ($\alpha < \beta$).

So $f(x) \leq 0$ for all $x \in (-\infty, \alpha] \cup [\beta, \infty)$ and $f(x) > 0$ for all $x \in (\alpha, \beta)$.

i.e. $f(x)$ is positive for some values of x and negative for other values of x .

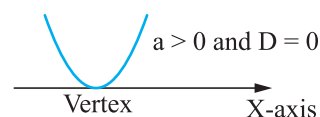

 5. $a > 0, D = 0$:

As $a > 0$, parabola opens upward.

As $D = 0$, parabola touches X-axis.

So $f(x) \geq 0$ for all $x \in R$.

i.e. $f(x)$ is positive for all values of x except at vertex where $f(x) = 0$.

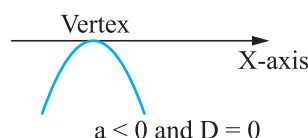

 6. $a < 0, D = 0$:

As $a < 0$, parabola opens downward.

As $D = 0$, parabola touches X-axis.

So $f(x) \leq 0$ for all $x \in R$.

i.e. $f(x)$ is negative for all values of x except at vertex where $f(x) = 0$.



4.4 Quadratic Inequation :

Let $f(x) = ax^2 + bx + c$ where $a, b, c \in R$ and $a \neq 0$. To solve the inequations of type :

$\{f(x) \leq 0 ; f(x) < 0 ; f(x) \geq 0 ; f(x) > 0\}$, we use the following procedure.

 (a) $D > 0$

➤ Make the coefficient of x^2 positive

➤ Factorise the expression and represent the left hand side of inequality in the form $(x - \alpha)(x - \beta)$.

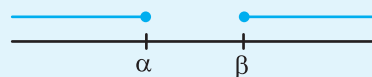
➤ If $(x - \alpha)(x - \beta) > 0$, then x lies outside α and β .

$$\Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$$



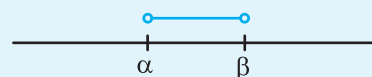
➤ If $(x - \alpha)(x - \beta) \geq 0$, then x lies on and outside α and β .

$$\Rightarrow x \in (-\infty, \alpha] \cup [\beta, \infty)$$



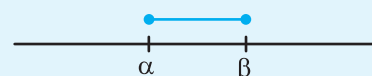
➤ If $(x - \alpha)(x - \beta) < 0$, then x lies inside α and β .

$$\Rightarrow x \in (\alpha, \beta)$$



➤ If $(x - \alpha)(x - \beta) \leq 0$, then x lies on and inside α and β .

$$\Rightarrow x \in [\alpha, \beta]$$



(b) $D < 0$ and $a > 0$: $f(x) > 0$ for all $x \in R$.

(c) $D < 0$ and $a < 0$: $f(x) < 0$ for all $x \in R$.

(d) $D = 0$ and $a > 0$: $f(x) \geq 0$ for all $x \in R$.

(e) $D = 0$ and $a < 0$: $f(x) \leq 0$ for all $x \in R$.

(f) $D \leq 0, a > 0$: $f(x) \geq 0$ for all $x \in R$.

(g) $D \leq 0, a < 0$: $f(x) \leq 0$ for all $x \in R$.

Illustrating the Concept :

(a) If $f(x) = x^2 + 2x + 2$, then solve the following inequalities :

(i) $f(x) \geq 0$

(ii) $f(x) \leq 0$

(iii) $f(x) > 0$

(iv) $f(x) < 0$

$f(x) = x^2 + 2x + 2$

Let us find D

$\Rightarrow D = b^2 - 4ac = (2)^2 - 4(1)(2) = -4 < 0 \Rightarrow D < 0$

$\Rightarrow$ roots of the corresponding equation ($f(x) = 0$) are non real.

$f(x)$ cannot be factorized into linear factor.

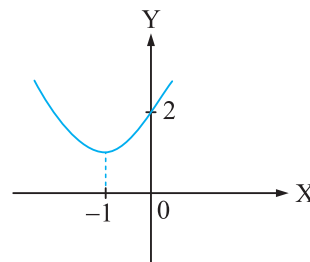
Also observe that $a = \text{coefficient of } x^2 = 1 > 0$

As $a > 0$ and $D < 0$, we get : $f(x) > 0 \forall x \in R$ [Using result 3.5 (b)]

(i) $f(x) \geq 0$ is true $\forall x \in R$ (ii) $f(x) \leq 0$ is true for no value of x i.e., $x \in \{ \}$.

(iii) $f(x) > 0$ is true $\forall x \in R$. (iv) $f(x) < 0$ is true for no value of x i.e., $x \in \{ \}$.

[Using results given in section 3.5, (b) and (c)]



(b) If $f(x) = x^2 + 4x + 4$, then solve the following inequalities :

(i) $f(x) \geq 0$

(ii) $f(x) \leq 0$

(iii) $f(x) > 0$

(iv) $f(x) < 0$

$f(x) = x^2 + 4x + 4 = (x + 2)^2$

Let us find D

$\Rightarrow D = b^2 - 4ac = (4)^2 - 4(1)(4) = 0 \Rightarrow D = 0$

$\Rightarrow$ roots of the corresponding equation ($f(x) = 0$) are real and equal.

Also observe that $a = \text{coefficient of } x^2 = 1 > 0$

As $D = 0$ and $a > 0$, we get :

$\Rightarrow f(x) \geq 0 \forall x \in R$ [Using 3.5 (d)]

(i) $f(x) \geq 0$ is true $\forall x \in R$. (ii) $f(x) \leq 0$ is true $\forall x \in \{-2\}$

(iii) $f(x) > 0$ is true $\forall x \in R - \{-2\}$ (iv) $f(x) < 0$ is true for no value of x i.e., $x \in \{ \}$.

[Using results given in section 3.5(c)]

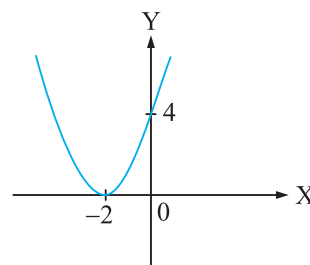


Illustration - 7 Solve the following quadratic inequality : $x^2 - 2x - 3 < 0$.

(A) $x \in [-1, 3]$

(B) $x \in (-1, 3)$

(C) $x \in [3, 4]$

(D) $(-\infty, -1) \cup (3, \infty)$

SOLUTION : (B)

$x^2 - 2x - 3 < 0$

Let us find D

$\Rightarrow D = b^2 - 4ac = (-2)^2 - 4(1)(-3) = 16 > 0 \Rightarrow D > 0$

Now factorize LHS using 'Splitting the middle term' method i.e.,

$x^2 - 3x + x - 3 < 0 \Rightarrow (x - 3)(x + 1) < 0$

$\Rightarrow x \in (-1, 3)$

[Using result mentioned in section 3.5 (a)]

Illustration - 8 Solve the following quadratic inequality : $x^2 + x - 1 \geq 0$

- (A) $x \in \mathbb{R}$ (B) $x \in \left[\frac{-1-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2} \right]$
 (C) $x \in \left(-\infty, \frac{-1-\sqrt{5}}{2} \right] \cup \left[\frac{\sqrt{5}-1}{2}, \infty \right)$ (D) No real value of x

SOLUTION : (C)

$$x^2 + x - 1 \geq 0 \quad \dots\dots(i)$$

Let us find D

$$\Rightarrow D = b^2 - 4ac = 1^2 - 4(1)(-1) = 5 > 0 \Rightarrow D > 0.$$

LHS cannot be factorized using splitting the middle term method. We will find roots of the corresponding equation (say α and β) then use result $ax^2 + bx + c = a(x - \alpha)(x - \beta)$, where α and β are roots of $ax^2 + bx + c = 0$. $\dots\dots(ii)$

$$\text{Consider } x^2 + x - 1 = 0$$

$$\text{Using } x = \frac{-b \pm \sqrt{D}}{2a} \text{ formula to find roots, we get } x = \frac{-1 \pm \sqrt{5}}{2}$$

Using (i), we get :

$$x^2 + x - 1 = \left[x - \left(\frac{-1+\sqrt{5}}{2} \right) \right] \left[x - \left(\frac{-1-\sqrt{5}}{2} \right) \right] \quad \dots\dots(iii)$$

Combining (i) and (iii), we get :

$$x^2 + x - 1 = \left[x - \left(\frac{-1+\sqrt{5}}{2} \right) \right] \left[x - \left(\frac{-1-\sqrt{5}}{2} \right) \right] \geq 0$$

$$x \in \left(-\infty, \frac{-1-\sqrt{5}}{2} \right] \cup \left[\frac{\sqrt{5}-1}{2}, \infty \right) \quad [\text{Using result given in section 3.5 (a)}]$$

4.5 Maximum and Minimum values of a Quadratic Polynomial

$$\text{Let } f(x) = ax^2 + bx + c, a \neq 0.$$

Case : I ($a > 0$) When $a > 0$, parabola opens upward.

From graph, vertex (V) is the lowest point on the graph.

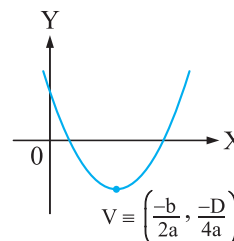
$$\Rightarrow y = f(x) \text{ possesses minimum value at } x = \frac{-b}{2a}.$$

$$\Rightarrow y_{\min} = f(x)_{\min} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a}$$

As you can observe on graph,

Maximum value of $f(x)$ is approaching to infinitely large value,

$$\text{i.e., } y_{\max} = f(x)_{\max} = \infty \text{ (not defined).}$$



Case : II ($a < 0$)

When $a < 0$, parabola opens downward.

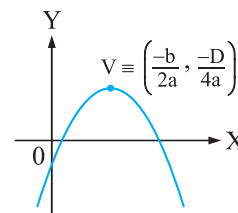
From graph, vertex (V) is the highest point on the graph.

$$y = f(x) \text{ possesses maximum value at } x = \frac{-b}{2a}$$

$$\Rightarrow y_{\max} = f(x)_{\max} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a}$$

As you can observe from graph, minimum value of $f(x)$ is approaching to infinitely small value.

i.e., $y_{\min} = f(x)_{\min} = -\infty$ (not defined).

**Illustrating the Concept :**

Find $f(\max)$ or $f(\min)$ in the following polynomials over $x \in R$.

(i) $f(x) = 4x^2 - 12x + 15$ (ii) $f(x) = -3x^2 + 5x - 4$

(i) $f(x) = 4x^2 - 12x + 15$

As $a = 4 > 0$

$f(x)$ has minimum value at vertex.

$$D = (-12)^2 - 4 \times 4 \times 15 = 144 - 240 = -96$$

$$f_{\min} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a} \Rightarrow f_{\min} = \frac{-(-96)}{4 \times 4} = \frac{96}{16} = 6 \text{ at } x = -\frac{-12}{2 \times 4} = \frac{3}{2}$$

$$\therefore f_{\min} = 6 \text{ at } x = \frac{3}{2}$$

$$f_{\max} = \infty$$

(ii) $f(x) = -3x^2 + 5x - 4$

As $a = -3 < 0$

$f(x)$ has maximum value at vertex.

$$D = (5)^2 - 4(-3)(-4) = 25 - 48 = -23$$

$$f_{\max} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a}$$

$$f_{\max} = -\frac{(-23)}{4(-3)} = -\frac{23}{12} \text{ at } x = \frac{-5}{2(-3)} = \frac{5}{6}$$

$$\therefore f_{\max} = \frac{-23}{12} \text{ at } x = \frac{5}{6}$$

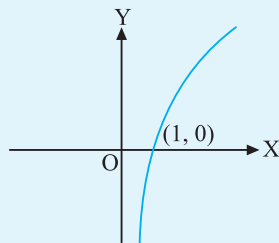
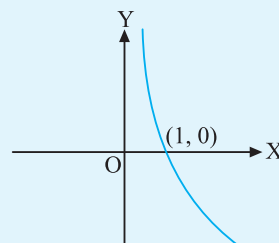
$$f_{\min} = -\infty$$

4.6 Introduction to Logarithmic Function :

If $x = a^y$ for $x > 0$, $a > 0$ and $a \neq 1$, then logarithm for x with respect to the base a is defined to be y . In symbols, $\log_a x = y$.

Note : If any of the conditions viz. (i) $x > 0$ (ii) $a > 0$ and (iii) $a \neq 1$ is not fulfilled, logarithm is invalid.

Graph of Logarithmic Function

$y = \log_a x, a > 1$ or $x = a^y; a > 1$	
(i) When $0 < x < 1$, $x = a^y$ We have to choose those values of y for which $0 < a^y < 1$. Since $a > 1, y < 0 \Rightarrow y \in (-\infty, 0)$. (ii) When $x = 1$, $x = a^y$ We have to choose that value of y for which x becomes 1 $\Rightarrow y = 0$. (iii) When $x > 1$, $x = a^y$ We have to choose those values of y for which $x > 1$. Since $a > 1, 0 < y < \infty$.	 Graph of $\log_a x, a > 1$
$y = \log_a x, 0 < a < 1$ or $x = a^y, 0 < a < 1$	
(iv) When $0 < x < 1$, We have to choose those values of y for which $0 < a^y < 1$. Since $0 < a < 1, y > 0$. (v) When $x = 1$, $x = a^y$ We have to choose that value of y for which x becomes 1 $\Rightarrow y = 0$ (vi) When $x > 1$, We have to choose those values of y for which $a^y > 1$. Since $a < 1, y < 0$.	 Graph of $\log_a x, 0 < a < 1$

4.7 General Properties of logarithmic function :

- | | |
|--|---|
| (i) $\log_a(xy) = \log_a x + \log_a y$; $x, y > 0, a > 0, a \neq 1$ | (ii) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$; $x, y > 0, a > 0, a \neq 1$ |
| (iii) $\log_a x^y = y \log_a x$; $x, a > 0, a \neq 1$ | (iv) $\log_{a^n} x = \frac{1}{n} \log_a x$; $x, a > 0, a \neq 1, n \neq 0$ |
| (v) $\log_{a^{2n}} x = \frac{1}{2n} \log_{ a } x$; $x > 0, a \neq -1, 0, 1, n \neq 0$ | (vi) $\log_a 1 = 0$; $a > 0, a \neq 1$ |
| (vii) $\log_a a = 1$; $a > 0, a \neq 1$ | (viii) $\log_y x = \frac{1}{\log_x y}$, where $x, y > 0, x \neq 1, y \neq 1$ |
| (ix) $\log_y x = \frac{\log_z x}{\log_z y}$, where $x, y, z > 0; y \neq 1, z \neq 1$ | (x) $x^{\log_a y} = y^{\log_a x}$; $x, y, a > 0, a \neq 1$ |
| (xi) $a^{\log_a x} = x$; $a, x > 0, a \neq 1$ | (xii) If $a > 1$, and $m > n > 0 \Leftrightarrow \log_a m > \log_a n$ |
| (xiii) If $0 < a < 1$, then $m > n > 0 \Leftrightarrow \log_a m < \log_a n$. | (xiv) If $a > 1$ then $\log_a x < p \Rightarrow 0 < x < a^p$ |

(xv) If $a > 1$ then $\log_a x > p \Rightarrow x > a^p$

(xvi) If $0 < a < 1$ then $\log_a x < p \Rightarrow x > a^p$

(xvii) If $0 < a < 1$ then $\log_a x > p \Rightarrow 0 < x < a^p$

(xviii) $e^{\ln f(x)} = f(x), f(x) > 0$

Note: Whenever the base of a logarithmic term is not written, its base is assumed to be 10. Logarithm of x to the base e is usually written as $\ln x$.

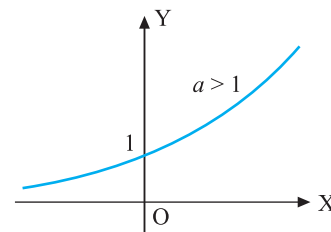
(b) Exponential Function

$y = a^x$ where $a > 1$ or $0 < a < 1$ is an exponential function of x .

This function is the inverse of logarithmic function *i.e.* it can be obtained by interchanging x and y in $y = \log_a x$.

As observed from the graph, if $a > 1$, then y increases as x increases.

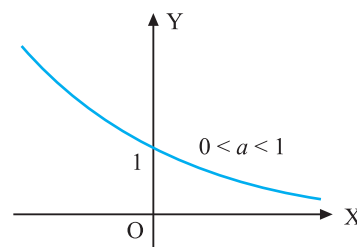
If $0 < a < 1$, then y decreases as x increases.

**Continuity :**

The graph of $f(x) = a^x$ is continuous (*i.e.* no break in the curve) everywhere.

Domain and Range :

The domain of the function $f(x)$ is $x \in R$ and Range is $y > 0$.

**Illustrating the Concepts :**

Prove that : $\left(\sqrt[3]{9}\right)^{\frac{1}{5\log_5 3}} = 25^{\frac{1}{15}}$

$$\begin{aligned} \text{L.H.S.} &= \left[\sqrt[3]{9}\right]^{\frac{1}{5\log_5 3}} \\ &= \left[9^{1/3}\right]^{\frac{1}{5\log_3 5}} = \left[\left(3^2\right)^{1/3}\right]^{\frac{1}{5\log_3 5}} = \left[3^{2/3}\right]^{\frac{1}{5\log_3 5}} \\ &= \left(3\right)^{\frac{2}{15\log_3 5}} = 3^{\frac{2}{\log_3 5}} = 5^{\frac{2}{15}} = \left(5^2\right)^{\frac{1}{15}} = 25^{\frac{1}{15}} = \text{R.H.S.} \end{aligned}$$

Illustration - 9

If $a > 0$, $a \neq 1$, then the equation $2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0$ has

(A) exactly one real root

(B) two real roots

(C) no real roots

(D) infinite number of real roots

SOLUTION : (B)

The equation can be written as

$$\frac{2\log a}{\log x} + \frac{\log a}{\log(ax)} + \frac{3\log a}{\log(a^2x)} = 0$$

.....(i)

$$\left[\text{Using } \log_a b = \frac{\log b}{\log a} \right]$$

As $a > 0$ and $a \neq 1$, $\log a \neq 0$, (i) can be written as

$$\frac{2}{y} + \frac{1}{b+y} + \frac{3}{2b+y} = 0$$

(where $b = \log a$ and $y = \log x$)

$$\Rightarrow 2(b+y)(2b+y) + y(2b+y) + 3y(b+y) = 0 \Rightarrow 4b^2 + 11by + 6y^2 = 0$$

Above equation is a quadratic in y . On solving, we get :

$$\Rightarrow y = \frac{-11b \pm \sqrt{121b^2 - 96b^2}}{12} = -\frac{4b}{3}, -\frac{b}{2}$$

As $y = \log x$ and $b = \log a$

$$\Rightarrow \log x = -\frac{4}{3} \log a \quad \text{or} \quad -\frac{1}{2} \log a \Rightarrow x = a^{-4/3}, a^{-1/2}$$

[Using : $\log_a b = c \Rightarrow b = a^c$]

$\Rightarrow$ Two real roots can exist.

Illustration - 10

If the graph of the polynomial : $y = x^2 + kx - x + 9$ is above X -axis, then the possible values of k are:

- (A) $k \in \mathbb{R}$ (B) $k \in (-5, 7)$ (C) $k \in (7, 9)$ (D) No real value of k

SOLUTION : (B)

$y = ax^2 + bx + c$ has its graph above x -axis if :

$a > 0$ and $D < 0$

Given $y = x^2 + (k-1)x + 9$

Coefficient of $x^2 = a = 1$ i.e. positive.

$D = (k-1)^2 - 36 < 0$, for graph to lie above x -axis

$$(k-7)(k+5) < 0$$

$\Rightarrow -5 < k < 7$ {For graph to lie above x -axis}

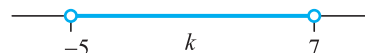


Illustration - 11

If $\log_2 (ax^2 + x + a) \geq 1 \quad \forall x \in \mathbb{R}$, then exhaustive set of values of 'a' is :

- (A) $\left(0, 1 + \frac{\sqrt{5}}{2}\right)$ (B) $\left(1 - \frac{\sqrt{5}}{2}, 1 + \frac{\sqrt{5}}{2}\right)$ (C) $\left(0, 1 - \frac{\sqrt{5}}{2}\right)$ (D) $\left[1 + \frac{\sqrt{5}}{2}, \infty\right)$

SOLUTION : (D)

$$\log_2 (ax^2 + x + a) \geq 1 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow ax^2 + x + a \geq 2 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow ax^2 + x + (a-2) \geq 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \text{coefficient of } x^2 > 0 \text{ and } D \leq 0$$

$$\Rightarrow a > 0 \quad \text{and} \quad 1 - 4a(a-2) \leq 0$$

$$\Rightarrow a > 0 \quad \text{and} \quad 4a^2 - 8a - 1 \geq 0$$

$$\Rightarrow a > 0 \quad \text{and} \quad a \in \left(-\infty, 1 - \frac{\sqrt{5}}{2}\right] \cup \left[1 + \frac{\sqrt{5}}{2}, \infty\right)$$

$$\Rightarrow a \in \left[1 + \frac{\sqrt{5}}{2}, \infty\right)$$

Illustration - 12

The least integral value of 'k' for which $(k-2)x^2 + 8x + k + 4 > 0$ for all $x \in \mathbb{R}$, is :

- (A) 5 (B) 2 (C) 3 (D) None of these

SOLUTION : (A)

$$\text{Let } f(x) = (k-2)x^2 + 8x + k + 4$$

$$f(x) > 0 \Rightarrow a > 0 \quad \text{and} \quad D < 0$$

$$k-2 > 0 \quad \text{and} \quad 64 - 4(k-2)(k+4) < 0$$

$$k > 2 \quad \text{and} \quad 16 - (k^2 + 2k - 8) < 0$$

$$k > 2 \quad \text{and} \quad k^2 + 2k - 24 > 0$$

$$k > 2 \quad \text{and} \quad (k < -6 \text{ or } k > 4)$$

As it can be observed that k can take values greater than 4

$$\Rightarrow k > 4.$$

$$\therefore \text{least integral value of } k = 5.$$

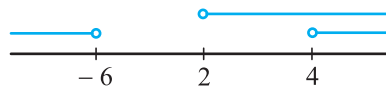


Illustration - 13 If $a < b$, then solution of $x^2 + (a + b)x + ab < 0$ is given by

- (A) $x < b$ or $x < a$ (B) $a < x < b$ (C) $x < a$ or $x > b$ (D) $-b < x < -a$

SOLUTION : (D)

$$x^2 + (a + b)x + ab < 0 \quad \Rightarrow \quad (x + a)(x + b) < 0 \quad \Rightarrow \quad -b < x < -a$$

Illustration - 14 If $a, b, c \in \mathbb{R}$ and $(a + b + c)c < 0$, then the quadratic equation $p(x) = ax^2 + bx + c = 0$ has :

- (A) A negative root (B) Two real roots (C) Two imaginary roots (D) None of these

SOLUTION : (B)

$$p(x) = ax^2 + bx + c = 0$$

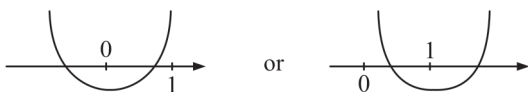
$$\text{Now, } a + b + c = p(1) \text{ and } c = p(0)$$

According to question,

$$(a + b + c)c < 0 \quad \Rightarrow \quad p(1)p(0) < 0$$

Possibility of graph :

1. $a > 0$:



$$\Rightarrow p(x) = 0 \text{ has one root in } (0, 1).$$

$p(x) = 0$ has two real roots because if coefficients and any one root are real, then other root would also be real.

2. $a < 0$:

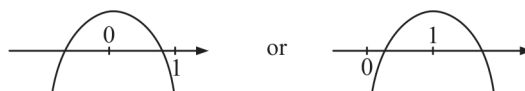


Illustration - 15 Let a, b, c be three distinct real numbers such that each of the expressions $ax^2 + bx + c$, $bx^2 + cx + a$ and

$cx^2 + ax + b$ are positive for all $x \in \mathbb{R}$ and let $\alpha = \frac{bc + ca + ab}{a^2 + b^2 + c^2}$ then

- (A) $\alpha < 4$ (B) $\alpha < 1$ (C) $\alpha > 1/4$ (D) $\alpha > 1$

SOLUTION : (B) & (C)

According to the given conditions $a > 0, b^2 < 4ac$; $b > 0, c^2 < 4ab$; $c > 0, a^2 < 4bc$

$$\therefore a^2 + b^2 + c^2 < 4(bc + ca + ab) \quad \dots\dots (i)$$

$$\Rightarrow \frac{1}{4} < \frac{bc + ca + ab}{a^2 + b^2 + c^2} \Rightarrow \frac{1}{4} < \alpha.$$

Also $a^2 + b^2 + c^2 - (bc + ca + ab) = \frac{1}{2} [(b-c)^2 + (c-a)^2 + (a-b)^2] \dots\dots(ii)$

But $\frac{1}{2} [(b-c)^2 + (c-a)^2 + (a-b)^2] > 0 \Rightarrow a^2 + b^2 + c^2 - (bc + ca + ab) > 0$ [Using (ii)]

$$\Rightarrow \frac{bc + ca + ab}{a^2 + b^2 + c^2} < 1 \Rightarrow \alpha < 1.$$

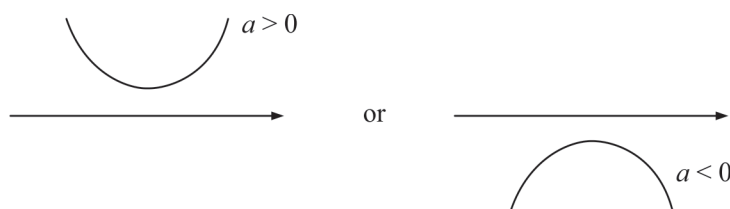
Illustration - 16 $a, b, c \in R, a \neq 0$ and the quadratic equation $ax^2 + bx + c = 0$ has no real roots, then

- (A) $a + b + c > 0$ (B) $a(a + b + c) > 0$ (C) $b(a + b + c) > 0$ (D) $c(a + b + c) > 0$

SOLUTION : (B) & (D)

Let $f(x) = ax^2 + bx + c$. It is given that $f(x) = 0$ has no real roots. So, either $f(x) > 0$ for all $x \in R$ or $f(x) < 0$ for all $x \in R$ i.e. $f(x)$ has same sign for all values of x .

Possibility of graph :



$$\therefore f(0)f(1) > 0 \Rightarrow c(a + b + c) > 0.$$

$$\text{Also, } af(1) > 0 \Rightarrow a(a + b + c) > 0.$$

Illustration - 17 Let $f(x) = x^2 + 4x + 1$, then

- (A) $f(x) > 0$ for all x (B) $f(x) \geq 1$ when $x \geq 0$
(C) $f(x) \geq 1$ when $x \leq -4$ (D) $f(x) = f(-x)$ for all x

SOLUTION : (B) & (C)

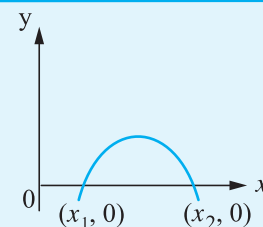
Since $f(x)$ is a quadratic expression having real roots ($\therefore D > 0$). Therefore $f(x)$ does not have the same sign for all x .

$$\begin{aligned} \text{Now, } f(x) \geq 1 &\Rightarrow x^2 + 4x + 1 \geq 1 &\Rightarrow x^2 + 4x \geq 0 \\ &\Rightarrow x \leq -4 \text{ or } x \geq 0 &\Rightarrow \text{(B) and (C) are correct.} \end{aligned}$$

$$f(-x) = x^2 - 4x + 1 \Rightarrow f(-x) \neq f(x) \Rightarrow \text{(D) is wrong}$$

Illustration - 18 The adjoining figure shows the graph of $y = ax^2 + bx + c$. Then

- (A) $a < 0$
(B) $b^2 < 4ac$
(C) $c > 0$
(D) a and b are of opposite signs.



SOLUTION : (A) & (D)

As it is clear from the figure that it is a parabola opening downwards i.e. $a < 0$. $\Rightarrow$ (A) is correct.

$\Rightarrow$ It is $y = ax^2 + bx + c$ i.e. degree two polynomial.

Now, if $ax^2 + bx + c = 0 \Rightarrow$ it has two roots x_1 and x_2 , as it cuts the axis at two distinct point x_1 and x_2 .

Now, from the figure it is also clear that $x_1 + x_2 > 0$. (i.e. sum of roots is positive)

$\Rightarrow \frac{-b}{a} > 0 \Rightarrow \frac{b}{a} < 0 \Rightarrow a$ and b are of opposite signs. $\Rightarrow$ (D) is correct.

As $D > 0$ and $f(0) = c < 0$, both (B) and (C) are wrong.

Illustration - 19 The diagram shows the graph of $y = ax^2 + bx + c$. Then,

(A) $a > 0$

(B) $b < 0$

(C) $c > 0$

(D) $b^2 - 4ac = 0$

SOLUTION : (B) & (C)

As it is clear from the figure that it is a parabola opening downwards i.e. $a < 0$.

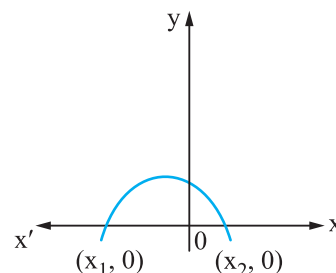
$\Rightarrow$ It is $y = ax^2 + bx + c$ i.e. degree two polynomial.

Now, if $ax^2 + bx + c = 0 \Rightarrow$ it has two roots x_1 and x_2 as it cuts the axis at two distinct point x_1 and x_2 .

Now from the figure it is also clear that $x_1 + x_2 < 0$. (i.e. sum of roots is negative)

$\Rightarrow \frac{-b}{a} < 0 \Rightarrow \frac{b}{a} > 0 \Rightarrow b < 0 \Rightarrow$ (B) is correct.

As the graph of $y = f(x)$ cuts the $+y$ -axis at $(0, c)$ where $c > 0 \Rightarrow$ (C) is correct.

**RATIONAL FUNCTION & RATIONAL INEQUATION****Section - 5****5.1 Introduction to Rational Functions**

Rational function of x is defined as ratio of two polynomials of x , say $P(x)$ and $Q(x)$ where $Q(x) \neq 0$ i.e.

$$\text{If } f(x) = \frac{P(x)}{Q(x)} ; Q(x) \neq 0,$$

then $f(x)$ is a rational function of x .

Following are some examples of rational functions of x .

$$f(x) = \frac{x+1}{x^2+x+1} ; f(x) = \frac{x^2-x+2}{x^2-5x+6} ; x \neq 2, x \neq 3 ; f(x) = \frac{x^4+x^3+x+1}{(x-1)^2} ; x \neq 1$$

5.2 Maximum and Minimum values of a Rational Function of x

Consider : $f(x) = y = \frac{ax^2 + bx + c}{px^2 + qx + r}$ where $x \in R - \{\alpha, \beta\}$, where α, β are roots of $px^2 + qx + r = 0$ (i)

$$x \in R - \{\alpha, \beta\} \rightarrow \boxed{\frac{ax^2 + bx + c}{px^2 + qx + r}} \rightarrow y \in ?$$

We will find maximum and minimum values $f(x)$ can take.

Cross Multiply in (i) to get :

$$y(px^2 + qx + r) = ax^2 + bx + c \quad \Rightarrow \quad (a - py)x^2 + (b - qy)x + (c - ry) = 0$$

As x is real, $D \geq 0$

$$\Rightarrow (b - qy)^2 - 4(a - py)(c - ry) \geq 0$$

Above relationship is an inequality in y . On solving the inequality we will get the values that y can take.

Case - I : $y \in [A, B]$

If y can take values between A and B , then,

Maximum value of $y = y_{\max} = B$, Minimum value of $y = y_{\min} = A$.

Case - II : $y \in (-\infty, A] \cup [B, \infty)$

If y can take values outside A and B , then

Maximum value of $y = y_{\max} = \infty$ i.e. not defined.

Minimum value of $y = y_{\min} = -\infty$ i.e. not defined.

Case - III : $y \in (-\infty, \infty)$ i.e. $y \in R$

If y can take all values, then

Maximum value of $y = y_{\max} = \infty$ i.e. not defined.

Minimum value of $y = y_{\min} = -\infty$ i.e. not defined.

Illustration - 20

If $f(x) = \frac{x^2 + 34x - 71}{x^2 + 2x - 7}$, $x \in R$, then $f(x)$ can take values :

- (A) (5, 9) (B) $(-\infty, 5] \cup [9, \infty)$ (C) [5, 9] (D) None of these

SOLUTION : (B)

$$\text{Let } \frac{x^2 + 34x - 71}{x^2 + 2x - 7} = k$$

By cross multiplying and making a quadratic equation in x , we get :

$$\Rightarrow x^2(1 - k) + (34 - 2k)x + 7k - 71 = 0$$

As $x \in R$, discriminant ≥ 0

$$\Rightarrow (34 - 2k)^2 - 4(1 - k)(7k - 71) \geq 0$$

$$\Rightarrow (17 - k)^2 - (1 - k)(7k - 71) \geq 0$$

$$\Rightarrow 8k^2 - 112k + 360 \geq 0$$

$$\Rightarrow k^2 - 14k + 45 \geq 0$$

$$\Rightarrow (k - 5)(k - 9) \geq 0$$

$$\Rightarrow k \in (-\infty, 5] \cup [9, \infty) \quad [\text{Using result 3.5 (a)}]$$

Hence k can never lie between 5 and 9.

Illustration - 21

The values of m for which the expression : $\frac{2x^2 - 5x + 3}{4x - m}$ can take all real values for $x \in R - \left\{\frac{m}{4}\right\}$

- (A) $m \in (4, 6)$ (B) $m \in [6, 8]$ (C) $m \in [-6, -4]$ (D) $m \in [-4, -2]$

SOLUTION : (A)

$$\text{Let } \frac{2x^2 - 5x + 3}{4x - m} = k \Rightarrow 2x^2 - (4k + 5)x + 3 + mk = 0$$

$$\Rightarrow \text{As } x \in R, \text{ discriminant} \geq 0$$

$$\Rightarrow (4k + 5)^2 - 4(3 + mk) \geq 0 \Rightarrow 16k^2 + (40 - 8m)k + 1 \geq 0$$

A quadratic in k is positive for all values of k if coefficient of k^2 is positive and discriminant ≤ 0 .

$$\Rightarrow (40 - 8m)^2 - 4(16)(1) \leq 0 \Rightarrow (5 - m)^2 - 1 \leq 0$$

$$\Rightarrow (m - 5 - 1)(m - 5 + 1) \leq 0 \Rightarrow (m - 6)(m - 4) \leq 0$$

$$\Rightarrow m \in [4, 6]$$

$$\text{Also } k = \frac{2x^2 - 5x + 3}{4x - m} = \frac{(x-1)(2x-3)}{(4x-m)}$$

$$\text{When } m = 4 \Rightarrow x \in R - \{1\} \quad \text{and} \quad k = \frac{(x-1)(2x-3)}{4(x-1)} = \frac{1}{4}(2x-3)$$

$$\text{Since } x \neq 1 \Rightarrow k \neq -\frac{1}{4}$$

$$\text{i.e. } k \text{ can't take all real value for } x \in R - \left\{\frac{m}{4}\right\}$$

$$\text{Similarly when } m = 6 \Rightarrow x \in R - \left\{\frac{3}{2}\right\} \quad \text{and} \quad k = \frac{(x-1)(2x-3)}{4x-6} = \frac{1}{2}(x-1)$$

$$\text{Since } x \neq \frac{3}{2} \Rightarrow k \neq \frac{1}{2}\left(\frac{3}{2} - 1\right) = \frac{1}{4}$$

$$\text{i.e. } k \text{ can't take all real value for } x \in R - \left\{\frac{m}{4}\right\}$$

Note : Here for $m = 4$ or 6 , given rational expression reduces to linear expression and linear expression can take all real values only when x can take all real values.

So for the given expression to take all real values, m should take values : $m \in (4, 6)$.

Illustration - 22

The values of m so that the inequality : $\left| \frac{x^2 + mx + 1}{x^2 + x + 1} \right| < 3$ holds for all $x \in R$.

- (A) $m \in (-1, 8)$ (B) $m \in (-\infty, -1) \cup (5, \infty)$ (C) $m \in (-1, 5)$ (D) None of these

SOLUTION : (C)

$$\text{We know that } |a| < b \Rightarrow -b < a < b$$

$$\text{Hence } \left| \frac{x^2 + mx + 1}{x^2 + x + 1} \right| < 3 \Rightarrow -3 < \frac{x^2 + mx + 1}{x^2 + x + 1} < 3$$

Case I : $\frac{x^2 + mx + 1}{x^2 + x + 1} < 3$

$$\Rightarrow \frac{(x^2 + mx + 1) - 3(x^2 + x + 1)}{x^2 + x + 1} < 0 \Rightarrow \frac{-2x^2 + (m-3)x - 2}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} < 0$$

Multiplying both sides by denominator, we get :

$$\Rightarrow -2x^2 + (m-3)x - 2 < 0 \quad (\text{because denominator is always positive})$$

$$\Rightarrow 2x^2 - (m-3)x + 2 > 0$$

A quadratic expression in x is always positive if coefficient of $x^2 > 0$ and $D < 0$.

$$\Rightarrow (m-3)^2 - 4(2)(2) < 0 \Rightarrow (m-3)^2 - 4^2 < 0 \Rightarrow (m-7)(m+1) < 0 \Rightarrow m \in (-1, 7) \quad \dots\dots(i)$$

Case II : $-3 < \frac{x^2 + mx + 1}{x^2 + x + 1} \Rightarrow 0 < \frac{(x^2 + mx + 1) + 3(x^2 + x + 1)}{x^2 + x + 1} \Rightarrow 4x^2 + (m+3)x + 4 > 0$

For this to be true for all $x \in R, D < 0$

$$\Rightarrow (m+3)^2 - 4(4)(4) < 0$$

$$\Rightarrow (m+3-8)(m+3+8) < 0 \quad [\text{Using } a^2 - b^2 = (a+b)(a-b)]$$

$$\Rightarrow (m-5)(m+11) < 0 \Rightarrow m \in (-11, 5) \quad \dots\dots(ii)$$

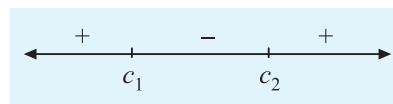
We will combine (i) and (ii) because both must be satisfied.

$$\Rightarrow \text{The common solution is } m \in (-1, 5).$$

5.3 Rational Algebraic Inequalities :

(a) Solving Quadratic Inequality :

A simple and quick method of solving quadratic inequations is as follows:



- Make the coefficient of x^2 positive if necessary.
- Check for $b^2 - 4ac$. If it is negative then the solution is either all real x or no real x depending on the inequality sign. If $b^2 - 4ac > 0$, then solve the given quadratic to get the real roots c_1 and c_2 where $c_1 < c_2$.
- If the final sign of the inequation is ' $>$ ' then the solution set is $(-\infty, c_1) \cup (c_2, \infty)$ and if the final sign is ' $<$ ' then the solution set is (c_1, c_2) .

(b) Equivalence in Inequalities :

The inequations are said to be equivalent if every solution of one is a solution of the other. For instance, the inequations

$$(x-2)(x-3) > 0 \text{ and } \frac{x-2}{x-3} > 0 \text{ are equivalent. The solution set to both inequations is } (-\infty, 2) \cup (3, \infty).$$

PROOF

The inequation $\frac{x-2}{x-3} > 0$ makes sense if $x \neq 3$. Multiplying by $(x-3)^2 > 0$ on both sides, we get $(x-2)(x-3) > 0$ whose solution set is easily seen to be $(-\infty, 2) \cup (3, \infty)$.

The student must note carefully that the inequations $(x-2)(x-3) \geq 0$ and $\frac{x-2}{x-3} \geq 0$ are NOT equivalent. The former has solution set $(-\infty, 2] \cup [3, \infty)$, while the latter has solution set $(-\infty, 2] \cup (3, \infty)$.

Similarly the inequations $(x-1)(x-2)(x-3) > 0$; $\frac{(x-1)(x-2)}{(x-3)} > 0$ and $\frac{(x-1)}{(x-2)(x-3)} > 0$

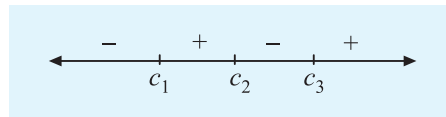
are all equivalent, (See the section for solutions to the cubic inequations)

(c) Cubic Inequalities :

Suppose the inequation can be written as $(x-c_1)(x-c_2)(x-c_3) > 0$, where $c_1 < c_2 < c_3$.

The solution set = $(c_1, c_2) \cup (c_3, \infty)$

If the sign in the above inequation is ' $<$ ', then the solution set is $(-\infty, c_1) \cup (c_2, c_3)$



(d) Generalization :

The solution set to the inequation $(x-c_1)(x-c_2) \dots (x-c_n) > 0$, where $c_1 < c_2 < \dots < c_n$ is

The solution set is : $(c_1, c_2) \cup (c_3, c_4) \cup \dots \cup (c_n, \infty)$, if n is odd

and $(-\infty, c_1) \cup (c_2, c_3) \cup \dots \cup (c_n, \infty)$, if n is even.

Note : Dealing with inequations in an immatured manner leads to serious errors. The consequences are generally

- (1) Allowing fake solutions (2) Discarding correct ones

(e) Rational Algebraic Inequalities :

Consider the following types of rational algebraic inequalities.

$$\frac{P(x)}{Q(x)} > 0, \frac{P(x)}{Q(x)} < 0, \frac{P(x)}{Q(x)} \geq 0, \frac{P(x)}{Q(x)} \leq 0$$

where $P(x)$ and $Q(x)$ are polynomials in x .

➤ These inequalities can be solved by the *method of intervals* also known as *sign method* or *wavy curve method*.

How to solve Rational Algebraic Inequality :

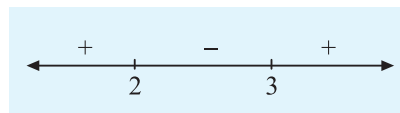
- Factorise $P(x)$ and $Q(x)$ into linear factors.
- Make coefficient of x positive in all factors.
- Equate all the factors to zero and find corresponding values of x . These values are known as critical points.
- Plot the critical points on a number line. n critical points will divide the number line $(n+1)$ regions.
- In right most region, the expression bears positive sign and in other regions the expression bears alternate positive and negative signs.

Illustrating the Concept :

- (i) Solve $x^2 - 5x + 6 > 0$.

It is easy to see that $x^2 - 5x + 6 = (x-2)(x-3)$.

Thus, the critical points are 2 and 3 and since the sign is ' $>$ ', the solution set is $(-\infty, 2) \cup (3, \infty)$

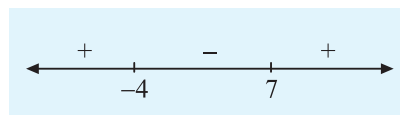


(ii) Solve $28 + 3x - x^2 > 0$

Multiplying with -1 , we get $x^2 - 3x - 28 < 0$

$$\Rightarrow (x + 4)(x - 7) < 0$$

$$\Rightarrow x \in (-4, 7)$$

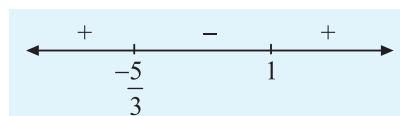


(iii) Solve $5 - 2x - 3x^2 \leq 0$.

Multiplying with -1 , we get $3x^2 + 2x - 5 \geq 0$.

$$\Rightarrow (x - 1)(3x + 5) \geq 0$$

$$\Rightarrow x \in \left(-\infty, -\frac{5}{3}\right] \cup [1, \infty)$$



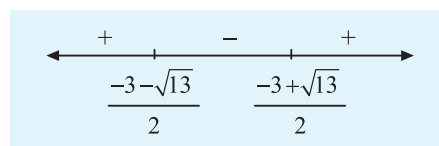
(iv) Solve $x^2 + 3x - 1 < 0$

Now the factors of $x^2 + 3x - 1$ are not possible by inspection although they are real.

Solving $x^2 + 3x - 1 = 0$, we get $\frac{-3 \pm \sqrt{13}}{2}$ as critical

points. Thus, the inequation can be written as

$$\left[x - \frac{-3 - \sqrt{13}}{2} \right] \left[x - \frac{-3 + \sqrt{13}}{2} \right] < 0$$

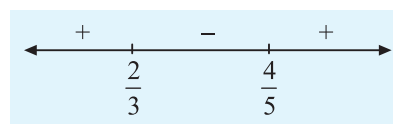


$$\text{Hence the solution set} = \left(\frac{-3 - \sqrt{13}}{2}, \frac{-3 + \sqrt{13}}{2} \right)$$

(v) Solve $\frac{3x-2}{5x-4} < 0$

The critical points are $2/3$ and $4/5$ and therefore the

solution is $\left(\frac{2}{3}, \frac{4}{5}\right)$

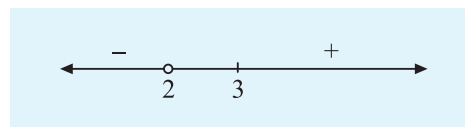


(vi) Solve $(x - 3)(x - 2)^2 > 0$.

Since $(x - 2)^2 > 0$ for all x except at $x = 2$ for which $(x - 2)^2 = 0$.

The given inequation is equivalent to $x - 3 > 0$ and $x \neq 2$.

Now $x - 3 > 0$ is obviously satisfied in $(3, \infty)$ which does not include 2. Thus the required solution is $(3, \infty)$



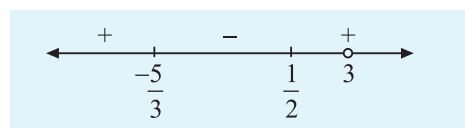
(vii) Solve $(x - 3)^2(2x - 1)(3x + 5) > 0$

Since $(x - 3)^2 > 0$ for all x except at $x = 3$ hence the inequation can be written as $(2x - 1)(3x + 5) > 0$ if $x \neq 3$.

$$\Rightarrow x \in \left(-\infty, -\frac{5}{3}\right) \cup \left(\frac{1}{2}, \infty\right), \text{ where } x \neq 3,$$

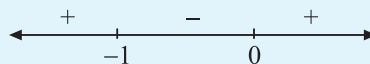
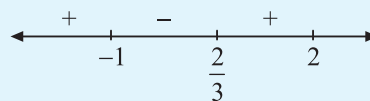
Hence, the required solution is :

$$\left(-\infty, -\frac{5}{3}\right) \cup \left(\frac{1}{2}, 3\right) \cup (3, \infty)$$



(viii) Solve $1/x > -1$

$$\frac{1}{x} + 1 > 0 \quad ; \quad \frac{x+1}{x} > 0$$

The critical points are -1 and 0 Hence, the required solution $= (-\infty, -1) \cup (0, \infty)$ (ix) Solve $(x-2)(3x-2)(x+1) < 0$.The critical points in the ascending order are -1 , $2/3$ and 2 using our algorithm for cubic inequality.Hence the solution is $(-\infty, -1) \cup \left(\frac{2}{3}, 2\right)$ **Illustration - 23**

Solve for x : $\frac{8x^2 + 16x - 51}{(2x-3)(x+4)} > 3$

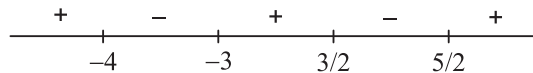
SOLUTION :

$$\begin{aligned} & \frac{8x^2 + 16x - 51}{(2x-3)(x+4)} > 3 \\ \Rightarrow & \frac{8x^2 + 16x - 51 - 3(2x-3)(x+4)}{(2x-3)(x+4)} > 0 \Rightarrow \frac{2x^2 + x - 15}{(2x-3)(x+4)} > 0 \Rightarrow \frac{(2x-5)(x+3)}{(2x-3)(x+4)} > 0 \end{aligned}$$

Critical points are : $x = -4, -3, 3/2, 5/2$

The solution from the number line is :

$$x \in (-\infty, -4) \cup \left(-3, \frac{3}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$$

**Illustration - 24**

Solve for x : $\frac{4}{1+x} + \frac{2}{1-x} < 1$.

SOLUTION :

$$\begin{aligned} & \frac{4}{1+x} + \frac{2}{1-x} < 1 \\ \text{On solving the above inequality, we get : } & \frac{x^2 - 2x + 5}{(1+x)(1-x)} < 0 \\ \Rightarrow & \frac{1}{(1+x)(1-x)} < 0 \quad \text{[As } x^2 - 2x + 5 > 0 \text{ for all } x \in \mathbb{R} \text{ (because } D < 0, a > 0)] \\ \Rightarrow & \frac{1}{(1+x)(x-1)} > 0 \Rightarrow x \in (-\infty, -1) \cup (1, \infty) \end{aligned}$$

Illustration - 25

Let $y = \sqrt{\frac{2}{x^2 - x + 1} - \frac{1}{x+1} - \frac{(2x+1)}{x^3 + 1}}$; the real values of x for which y takes real values (i.e. domain) are :

(A) $x \in (-1, 0) \cup (1, 2)$

(B) $x \in (-\infty, -1) \cup [0, 1]$

(C) $x \in (0, 1) \cup (2, 3)$

(D) $x \in (-1, 1)$

SOLUTION : (B)

For y to take real values ; $\frac{2}{x^2 - x + 1} - \frac{1}{x + 1} - \frac{(2x + 1)}{x^3 + 1} \geq 0$.

$$\Rightarrow \frac{2(x + 1) - (x^2 + 1 - x) - (2x + 1)}{x^3 + 1} \geq 0 \Rightarrow \frac{-x^2 + x}{(x + 1)(x^2 - x + 1)} \geq 0$$

$$\Rightarrow \frac{x(x - 1)}{(x + 1)(x^2 - x + 1)} \leq 0$$

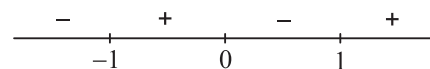
Multiply both sides by $x^2 - x + 1$ to get, [As $x^2 - x + 1 > 0$ for all $x \in R$ (because $D < 0$, $a > 0$), we can multiply both sides by $x^2 - x + 1$]

$$\Rightarrow \frac{x(x - 1)}{(x + 1)} \leq 0$$

Critical points are $x = 0, x = 1, x = -1$.

Expression is negative for $x \in (-\infty, -1) \cup [0, 1]$

So real values of x for which y is real are $x \in (-\infty, -1) \cup [0, 1]$.


Illustration - 26 Solve for x, if $|x - 1| \geq 3$.

SOLUTION :

Given $|x - 1| \geq 3$

$$\Rightarrow x - 1 \leq -3 \text{ or } x - 1 \geq 3$$

$$\Rightarrow x \leq -2 \text{ or } x \geq 4.$$

$$\text{Hence } x \in (-\infty, -2] \cup [4, \infty).$$

Illustration - 27 Solve $\sqrt{3x + 4} + \sqrt{x - 4} = 2\sqrt{x}$; $x \in R$
SOLUTION :

On squaring, we get :

$$3x + 4 + x - 4 + 2\sqrt{(3x + 4)(x - 4)} = 4x$$

On simplifying, we get $\sqrt{(3x + 4)(x - 4)} = 0$, whose roots are easily seen to be $x = -4/3$ and $x = 4$.

On verification, $x = -4/3$ does not satisfy the given equation.

Hence $x = 4$ is the only solution.

Illustration - 28 Solve $\sqrt{25 - x} = 2 - \sqrt{9 + x}$
SOLUTION :

Proceeding as earlier we get: $\sqrt{25 - x} + \sqrt{9 + x} = 2$

$$\Rightarrow 25 - x + 9 + x + 2\sqrt{(25 - x)(9 + x)} = 4$$

$$\Rightarrow -15 = \sqrt{(25 - x)(9 + x)} \quad \dots (i)$$

Squaring and simplifying, we get : $x^2 - 16x = 0$

$$\Rightarrow x = 0, 16$$

Both $x = 0$ and $x = 16$, do not satisfy the original equation. They appear because of squaring at the step (i).

Thus the given equation does not have any roots.

Illustration - 29 Solve $|3x - 2| = 1$ **SOLUTION :****Method 1 :** $|3x - 2| = 1 \Rightarrow 3x - 2 = \pm 1$ Considering positive sign, we get $x = 1$ Considering negative sign, we get $x = 1/3$.Thus the solution set is $\{1/3, 1\}$ **Method 2 :** Assume $3x - 2 \geq 0$ i.e., $x \geq 2/3$,
the equation becomes $3x - 2 = 1$ or $x = 1$.Now as $1 \geq 2/3$ is satisfied,Hence $x = 1$ is a root.Next, assume $3x - 2 < 0$, i.e., $x < 2/3$, the equation becomes $2 - 3x = 1$ or $x = 1/3$, which satisfies $1/3 < 2/3$.Hence $x = 1/3$ is also a root.

Thus the solution set is :

$$\{1/3, 1\}.$$

Note: Method 1 has its limitations. It can not be applied to all type of problems involving modulus sign. Hence we will be considering method 2 for most of the time.**Illustration - 30** Solve $|x - 3| = 7 - 2x$ **SOLUTION :**Over $[3, \infty)$, the equation is $x - 3 = 7 - 2x$ or $x = 10/3$, since $10/3 \geq 3$, it is a root.Over $(-\infty, 3)$, the equation becomes $3 - x = 7 - 2x$.

$$\Rightarrow x = 4 \text{ which does not satisfy } x < 3.$$

Thus the given equation has only one root i.e., $x = \frac{10}{3}$ **Illustration - 31** Solve $|x + 1| + 3x = |2x - 1| + 3$ **SOLUTION :**The critical points are -1 and $1/2$. Therefore we will divide R into three disjoint sets i.e.,

$$R = (-\infty, -1] \cup (-1, 1/2] \cup (1/2, \infty)$$

Over $(-\infty, -1]$, the equation is equivalent to

$$-x - 1 + 3x = -2x + 1 + 3$$

$$\Rightarrow x = 5/4, \text{ which does not lie in } (-\infty, -1].$$

Thus there is no root in $(-\infty, -1]$.Again over $(-1, 1/2]$, the equation is equivalent to

$$x + 1 + 3x = 1 - 2x + 3$$

$$\Rightarrow x = 1/2 \text{ is a solution as it belongs to } [-1, 1/2]$$

Finally over $(1/2, \infty)$, we get :

$$x + 1 + 3x = 2x - 1 + 3$$

$$\Rightarrow x = 1/2, \text{ which does not lie in } (1/2, \infty).$$

Thus there is no root in $(1/2, \infty)$

Thus the given equation has only one root

$$\text{i.e., } x = 1/2.$$

Illustration - 32 Solve $|3x - 2| > 1/2$.**SOLUTION :**

$$|x| \geq a \Rightarrow x \in (-\infty, -a] \cup [a, \infty) \dots (i)$$

If $a < 0$ then (i) is true for all real values of x .Now, following the above result, the given inequality implies $3x - 2 > 1/2$ or $3x - 2 < -1/2$

$$\Rightarrow 3x > 5/2 \text{ or } 3x < 3/2$$

$$\Rightarrow x > 5/6 \text{ or } x < 1/2$$

$$\Rightarrow x \in (-\infty, 1/2) \cup (5/6, \infty)$$

Illustration - 33 Solve $|2x + 5| \leq 1/2$

SOLUTION :

We must have $-1/2 \leq 2x + 5 \leq 1/2$

$$\Rightarrow -11/2 \leq 2x \leq -9/2$$

$$\Rightarrow -11/4 \leq x \leq -9/4$$

$$\Rightarrow x \in [-11/4, -9/4]$$

Illustration - 34 Solve $|3 - 7x| < -1$.

SOLUTION :

The expression $3 - 7x$ can be positive for some values of x and negative for some other values; but for all the values $|3 - 7x| \geq 0$.

Therefore the above inequation does not have real solutions.

(Note that $|3 - 7x| > -1$ is true for all real values of x)

Illustration - 35 Solve $x^2 - |x| - 12 < 0$.

SOLUTION :

Over $[0, \infty)$, the inequation is equivalent to

$$x^2 - x - 12 < 0.$$

$$\Rightarrow (x - 4)(x + 3) < 0 \Rightarrow x \in (-3, 4)$$

But as the equivalence is over $[0, \infty)$, the solution set becomes $[0, 4)$.

Again, over $(-\infty, 0)$, we have

$$x^2 + x - 12 < 0 \Rightarrow x \in (-4, 3)$$

But as the equivalence is over $(-\infty, 0)$, the solution set becomes $(-4, 0)$

Thus the solution to the given equation is :

$$(-4, 0) \cup [0, 4) = (-4, 4)$$

Illustration - 36 Solve $|x| - |x - 2| = 2$

SOLUTION :

Over $(-\infty, 0]$, we get $-x - (2 - x) = 2$

$$\Rightarrow -2 = 2 \text{ which is absurd}$$

Thus no solution in $(-\infty, 0]$

Over $(0, 2]$, we have

$$x - (2 - x) = 2 \quad \text{or} \quad x = 2$$

$$\Rightarrow x = 2 \text{ is a solution}$$

Over $(2, \infty)$, we get :

$$x - (x - 2) = 2 \quad \text{or} \quad 2 = 2$$

$$\Rightarrow \text{All real } x > 2 \text{ are the solutions.}$$

Hence the complete solution set is $[2, \infty)$.

Illustration - 37 Solve the following inequations.

(i) $\sqrt{x-3} > -2$ (ii) $\sqrt{x-3} > 2$ (iii) $\sqrt{x-3} < -2$ (iv) $\sqrt{x-3} < 2$

SOLUTION :

(i) The inequation makes sense if $x - 3 \geq 0$ i.e., $x \geq 3$.

Now as $\sqrt{x-3} \geq 0$ for all permissible x we must have $\sqrt{x-3} > -2$ for all permissible x . Thus the solution set is set of all those x 's for which the inequation makes sense, which is $[3, \infty)$.

- (ii) The domain D of the inequation $= [3, \infty)$, since both sides are non negative in the inequation, we can square both sides to get $x > 7$.
Hence the solution set $= (7, \infty) \cap [3, \infty) = (7, \infty)$
- (iii) Since $\sqrt{x-3} \geq 0$, for all permissible x the given inequation has no solution.
- (iv) Proceeding as in (ii), we get after squaring $x < 7$. Taking the intersection with $[3, \infty)$, we get the required solution set as $[3, 7)$.

Illustration - 38 Solve the following inequations.

(i) $\sqrt{\frac{x-2}{1-3x}} > -1$ (ii) $\sqrt{\frac{x-2}{1-3x}} > 1$ (iii) $\sqrt{\frac{x-2}{1-3x}} < -1$ (iv) $\sqrt{\frac{x-2}{1-3x}} < 1$

SOLUTION :

- (i) All those x 's for which the inequation has meaning will be solutions. The inequation makes sense

$$\text{if } \frac{x-2}{1-3x} \geq 0$$

$$\text{i.e. } \frac{x-2}{3x-1} \leq 0 \quad \Rightarrow \quad x \in (1/3, 2]$$

Hence the solution set $= (1/3, 2]$

- (ii) $D = (1/3, 2]$, where D denotes the domain of the function.

We square the inequation to get:

$$\begin{aligned} \frac{x-2}{1-3x} > 1 &\Rightarrow \frac{x-2}{1-3x} - 1 > 0 \\ \Rightarrow \frac{4x-3}{3x-1} < 0 &\Rightarrow x \in (1/3, 3/4) \end{aligned}$$

Taking the intersection with D , we get the required solution set as $(1/3, 3/4)$.

- (iii) Solution set $= \phi$, as LHS is non negative and RHS is negative

- (iv) $D = (1/3, 2]$ where D denotes the domain of the function.

Proceeding as in (ii), we get : $\frac{4x-3}{3x-1} > 0$

$$\Rightarrow x \in (-\infty, 1/3) \cup (3/4, \infty)$$

Taking intersection with D , we get the required solution set as $(3/4, 2]$.

Illustration - 39 Solve : (i) $x > \sqrt{24-5x}$ (ii) $x < \sqrt{24-5x}$

SOLUTION :

For all parts of the problem, $D = (-\infty, 24/5]$

- (i) For solving, inequation is to be squared. The squaring is possible if both sides are non-negative. The LHS ' x ' is negative in $(-\infty, 0)$ and positive in $(0, 24/5]$.

Let $D = D_1 \cup D_2$, where $D_1 = (-\infty, 0]$

and $D_2 = (0, 24/5]$

Now no value in D_1 is a solution since a negative quantity can not be greater than a non-negative quantity.

Thus, solution set S_1 over D_1 is ϕ .

Suppose $x \in D_2$, then on squaring we get :

$$\begin{aligned} x^2 &> 24 - 5x \\ \Rightarrow x^2 + 5x - 24 &> 0 \\ \Rightarrow (x+8)(x-3) &> 0 \\ \Rightarrow x &\in (-\infty, -8) \cup (3, \infty) \end{aligned}$$

On taking intersection with D_2 , we get :

$$S_2 = (3, 24/5]$$

Thus required solution set $= S_1 \cup S_2 = (3, 24/5]$

- (ii) Now, each value in D_1 is a solution.

Thus $S_1 = (-\infty, 0]$ on D_2 , we get :

$$x^2 < 24 - 5x \Rightarrow x \in (-8, 3)$$

Illustration - 40 Solve $\sqrt{x-5} - \sqrt{9-x} > 1$

SOLUTION :

Clearly, the domain of the function is $D = [5, 9]$.

Writing the given inequation as

$$\sqrt{x-5} > 1 + \sqrt{9-x} \quad \dots (i)$$

On squaring (both sides are non negative), and simplifying, we get :

$$2x - 15 > 2\sqrt{9-x} \quad \dots (ii)$$

Consider $D = [5, 15/2] \cup [15/2, 9]$, no value in $[5, 15/2]$ is a solution. Hence $S_1 = \phi$.

For $x \in [15/2, 9]$, we get :

$$(2x - 15)^2 > 4(9 - x)$$

$$\Rightarrow 4x^2 - 56x + 189 > 0$$

$$\Rightarrow x \in \left(-\infty, \frac{14-\sqrt{7}}{2}\right) \cup \left(\frac{14+\sqrt{7}}{2}, \infty\right)$$

Taking the intersection above interval with $[15/2, 9]$,

$$\text{we get : } S_2 = \left[\frac{14+\sqrt{7}}{2}, 9\right]$$

Thus the required solution set is :

$$\left[\frac{14+\sqrt{7}}{2}, 9\right]$$

Illustration - 41 Solve : $6 \cdot 4^x - 13 \cdot 6^x + 6 \cdot 9^x = 0$

SOLUTION :

On dividing by 6^x , we get :

$$6\left[\frac{4}{6}\right]^x - 13 + 6\left[\frac{3}{2}\right]^x = 0$$

$$\text{Put } \left[\frac{2}{3}\right]^x = t, \text{ we get :}$$

$$\Rightarrow 6t - 13 + \frac{6}{t} = 0 \Rightarrow 6t^2 - 13t + 6 = 0$$

$$\Rightarrow t = \frac{3}{2}, \frac{2}{3}$$

$$\Rightarrow \left[\frac{2}{3}\right]^x = \frac{3}{2} \text{ or } \frac{2}{3}$$

$$\Rightarrow x = 1 \text{ or } -1$$

Hence the solution set = $\{1, -1\}$

Illustration - 42 Solve $2^{2x^2} + 2^{x^2+2x+2} = 2^{5+4x}$

SOLUTION :

On dividing the given equation by 2^{4x} , we get :

$$= 2^{2x^2-4x} + 2^{x^2-2x+2} = 2^5$$

$$= 2^{2(x^2-2x)} + 4 \cdot 2^{x^2-2x} = 32$$

$$\text{Put } 2^{x^2-2x} = t, \text{ we get :}$$

$$t^2 + 4t - 32 = 0$$

$$\Rightarrow t = 4, -8$$

Now $t = -8$ is inadmissible since $2^{x^2-2x} > 0$ for all real x .

Therefore $t = 4$ is the only possibility.

$$\Rightarrow 2^{x^2-2x} = 4 = 2^2 \Rightarrow x^2 - 2x = 2$$

$$\Rightarrow x^2 - 2x - 2 = 0 \Rightarrow x = 1 \pm \sqrt{3}$$

Hence the solution set $S = \{1 - \sqrt{3}, 1 + \sqrt{3}\}$.

Illustration - 43 Solve $3^x + 5^x = 34$.**SOLUTION :**

We cannot solve this equation by standard methods. By hit and trial we note that $x = 2$ is a root. Let us show that a number different from 2 cannot be a solution of the above equation.

$$\text{Let } x_0 > 2 \text{ then } 3^{x_0} > 3^2 = 9$$

$$\text{and } 5^{x_0} > 5^2 = 25 = 3^{x_0} + 5^{x_0} > 34$$

A number greater than 2 cannot be solution of the given equation. We can similarly prove that a number smaller than 2 cannot be solution either.

Thus the only solution is $x = 2$.

Illustration - 44 Solve $2^x + 2^{-x+1} - 3 < 0$ **SOLUTION :**

$$\text{Put } 2^x = t, \text{ we get } t + \frac{2}{t} - 3 < 0$$

Multiplying by t , we get: (note that $t > 0$)

$$t^2 - 3t + 2 < 0 \quad \text{or} \quad (t-1)(t-2) < 0$$

$$\Rightarrow t \in (1, 2)$$

$$\Rightarrow 1 < 2^x < 2$$

On taking logs to base 2, we get :

$$\log 1 < x < \log 2$$

$$\Rightarrow 0 < x < 1$$

Note : From the inequality $a < c < b$ follows the inequality $\log a < \log c < \log b$ provided (i) $a, b, c > 0$ and (ii) $\log$ is taken at a base which is greater than unity.

Illustration - 45 Solve $x^2 5^x - 5^{x+2} < 0$ **SOLUTION :**

The given inequation can be written as

$$5^x (x^2 - 25) < 0 \quad \dots (i)$$

Now since $5^x > 0$ for all x , (i) is equivalent to

$$x^2 - 25 < 0 \Rightarrow x \in (-5, 5)$$

Illustration - 46 Solve $\frac{4^x + 2x - 4}{x-1} < 2$ **SOLUTION :**

$$\text{The given inequation is } \frac{4^x + 2x - 4}{x-1} - 2 < 0$$

$$\Rightarrow \frac{4^x - 2}{x-1} < 0$$

Case I : $4^x - 2 > 0$ and $x - 1 < 0$

$$\Rightarrow 4^x > 2 \quad \text{and} \quad x < 1$$

$$\Rightarrow 4^x > 4^{1/2} \quad \text{and} \quad x < 1$$

$$\Rightarrow x > 1/2 \quad \text{and} \quad x < 1$$

$$\Rightarrow x \in (1/2, 1)$$

Case II : $4^x - 2 < 0$ and $x - 1 > 0$

$$\Rightarrow x < 1/2 \quad \text{and} \quad x > 1$$

$\Rightarrow$ Hence no real value of x exists.

Thus the complete solution set to the given inequality is $(1/2, 1)$

Illustration - 47 Solve $\log_{3/2} \frac{2x-8}{x-2} < 0$

SOLUTION :

The inequation makes sense if $\frac{2x-8}{x-2} > 0$

$$\Rightarrow D = (-\infty, 2) \cup (4, \infty)$$

For these values, the given inequation is equivalent to

$$\Rightarrow \frac{2x-8}{x-2} < 1 \Rightarrow \frac{2x-8}{x-2} - 1 < 0$$

$$\Rightarrow \frac{x-6}{x-2} < 0 \Rightarrow x \in (2, 6)$$

Taking intersection with D , we get the required solution as $x \in (4, 6)$.

POSITION OF ROOTS OF A QUADRATIC EQUATION

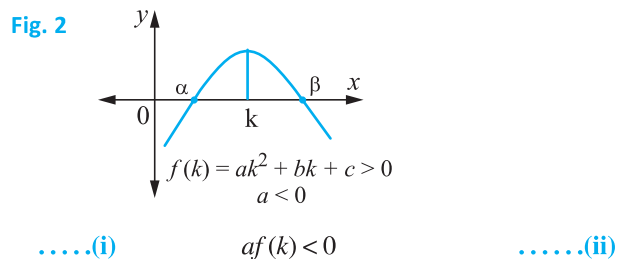
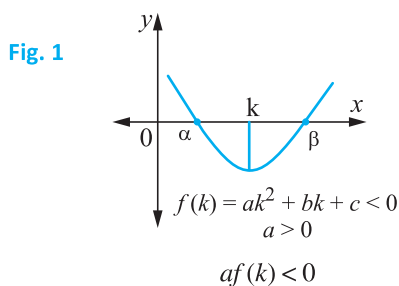
Section - 6

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ be a quadratic expression and k, k_1, k_2 be real numbers such that $k_1 < k_2$.

Let α, β ($\alpha \neq \beta$) be the roots of the equation $f(x) = 0$ i.e. $ax^2 + bx + c = 0$. Then $\alpha = \frac{-b - \sqrt{D}}{2a}$ and $\beta = \frac{-b + \sqrt{D}}{2a}$, where D is the discriminant of the equation.

6.1 Conditions for a number k to lie between the roots of a quadratic equation

If a number k lies between the roots of a quadratic equation $f(x) = ax^2 + bx + c = 0$, then the equation must have real roots and the sign of $f(k)$ is opposite to the sign of 'a' as is evident from Fig. 1 and 2.



Combining (i) and (ii), $af(k) < 0$ for k to lie between roots.

Note : For roots to be real, $D \geq 0$. There is no need to take this condition as when $af(k) < 0$, then D will always be positive i.e. $D \geq 0$. Hence $af(k) < 0$ is necessary and sufficient condition for k to lie between roots.

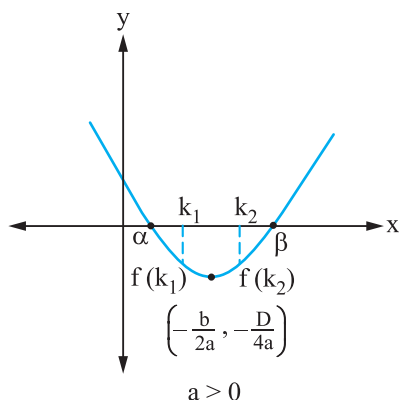
Thus, a number k lies between the roots of a quadratic equation $f(x) = ax^2 + bx + c = 0$

If $af(k) < 0$.

6.2 Conditions for both k_1 and k_2 to lie between the roots of a quadratic equation

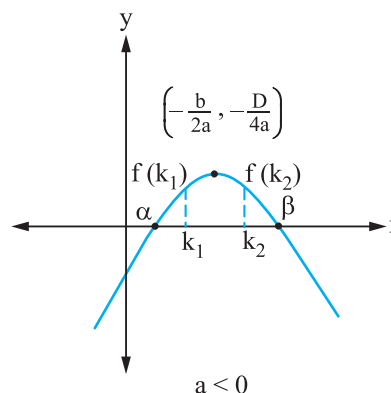
If both k_1 and k_2 lie between the roots α and β of a quadratic equation $f(x) = ax^2 + bx + c = 0$, then sign of $f(k_1)$ and $f(k_2)$ should be positive or negative depending upon sign of a as it is evident from figure 3 and 4.

Fig. 3



$$f(k_1) < 0 \text{ and } f(k_2) < 0 \quad \dots\dots(i)$$

Fig. 4



$$f(k_1) > 0 \text{ and } f(k_2) > 0 \quad \dots\dots(ii)$$

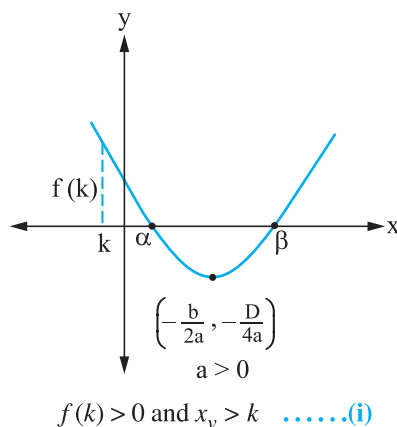
Combining (i) and (ii), $af(k_1) < 0$ and $af(k_2) < 0$

Hence for k_1 and k_2 to lie between roots, $af(k_1) < 0$ and $af(k_2) < 0$.

6.3 Conditions for a number k to be less than roots of a quadratic equation

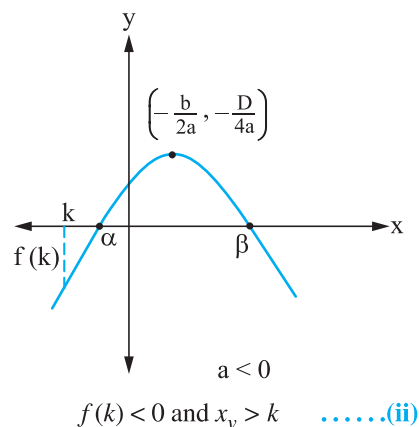
If a number k is smaller than the roots of a quadratic equation $f(x) = ax^2 + bx + c$, then the equation must have real and distinct roots and the sign of $f(k)$ is same as the sign of ' a ' as is evident from Figs. 5 and 6. Also, k is less than the x-coordinate of the vertex of the parabola $y = ax^2 + bx + c$ i.e. $k < -b/2a$.

Fig. 5



$$f(k) > 0 \text{ and } x_v > k \quad \dots\dots(i)$$

Fig. 6



$$f(k) < 0 \text{ and } x_v > k \quad \dots\dots(ii)$$

Combining (i) and (ii) we get : $af(k) > 0, x_v > k$ i.e. $-\frac{b}{2a} > k$ and $D \geq 0$

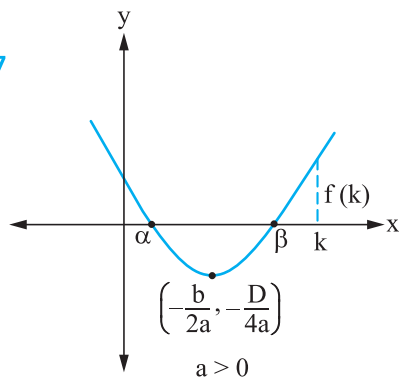
Thus, a number k is smaller than the roots of a quadratic equation $ax^2 + bx + c = 0$, if

$$(i) \quad D \geq 0 \quad (ii) \quad af(k) > 0 \quad (iii) \quad k < x_v = -b/2a.$$

6.4 Conditions for a number k to be more than the roots of a quadratic equation

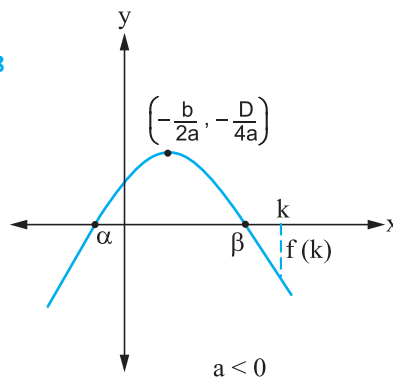
If a number k is larger than the roots of a quadratic equation $f(x) = ax^2 + bx + c$, then the equation must have real and distinct roots and the sign of $f(k)$ is same as the sign of ' a ' as is evident from Figs. 7 and 8. Also, k is greater than the x-coordinate of the vertex of the parabola $y = ax^2 + bx + c$ i.e. $k > -b/2a$.

Fig. 7



$$f(k) > 0 \text{ and } x_v < k \quad \dots\dots(i)$$

Fig. 8



$$f(k) < 0 \text{ and } x_v < k \quad \dots\dots(ii)$$

Combining (i) and (ii) we get : $af(k) > 0, x_v < k$ i.e. $-\frac{b}{2a} < k$ and $D \geq 0$

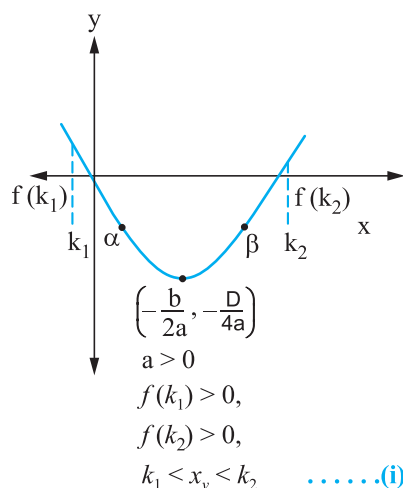
Thus, a number k is more than the roots of a quadratic equation $ax^2 + bx + c = 0$, if

$$(i) \quad D \geq 0 \quad (ii) \quad af(k) > 0 \quad (iii) \quad k > -b/2a.$$

6.5 Condition for both the roots of a quadratic equation to lie between numbers k_1 and k_2

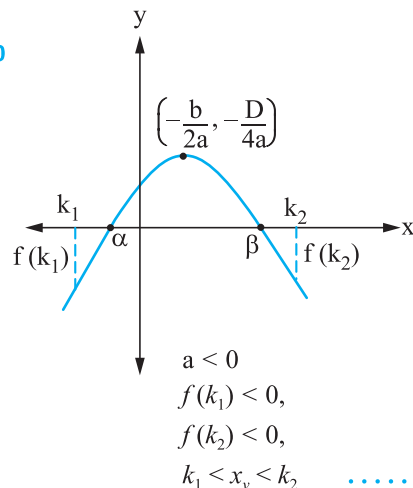
If both roots α and β of a quadratic equation $f(x) = ax^2 + bx + c = 0$, lie between numbers k_1 and k_2 , then equation must have real roots, signs of $f(k_1)$ and $f(k_2)$ are same as sign of ' a ' as is evident from fig. 9 and 10. Also $x_v = -\frac{b}{2a}$ must be between k_1 and k_2

Fig. 9



$$a > 0, \\ f(k_1) > 0, \\ f(k_2) > 0, \\ k_1 < x_v < k_2 \quad \dots\dots(i)$$

Fig. 10



$$a < 0, \\ f(k_1) < 0, \\ f(k_2) < 0, \\ k_1 < x_v < k_2 \quad \dots\dots(ii)$$

Combining (i) and (ii), we get : $af(k_1) > 0, af(k_2) > 0, k_1 < x_v < k_2$ and $D \geq 0$

If both the roots of a quadratic equation lie between numbers k_1 and k_2 , then

$$(i) \quad D \geq 0 \quad (ii) \quad af(k_1) > 0, af(k_2) > 0 \quad (iii) \quad k_1 < -\frac{b}{2a} < k_2.$$

6.6 Condition for exactly one root of a quadratic equation to lie in the interval (k_1, k_2) , where $k_1 < k_2$

If exactly one root of the equation $ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) , then $f(k_1)$ and $f(k_2)$ must be of opposite signs as shown in Figs. 11 and 12.

Fig. 11

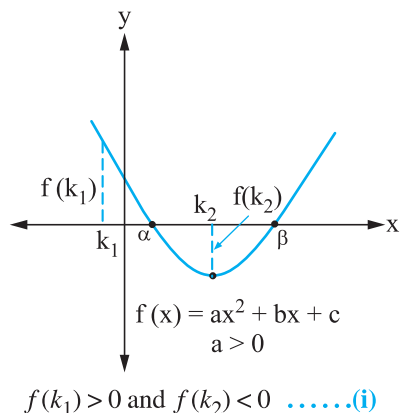
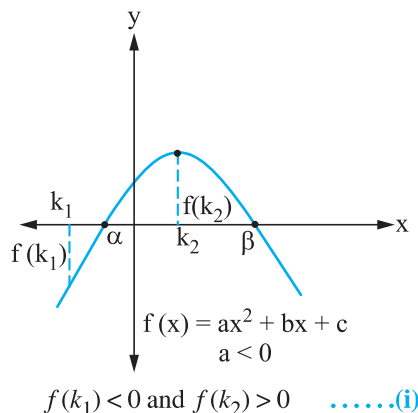


Fig. 12



Combining (i) and (ii), we get : $f(k_1)f(k_2) < 0$

Note : Exactly one root lies between k_1 and k_2 . Therefore graph of quadratic polynomial will cross x-axis only once between k_1 and k_2 . This implies signs of $f(k_1)$ and $f(k_2)$ would be different. Hence $f(k_1)f(k_2) < 0$

Thus, exactly one root of the equation $ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if

(i) $f(k_1)f(k_2) < 0$

6.7 Some More Result on Roots of Quadratic Equation

- Both roots of $f(x) = 0$ are negative,
if sum of the roots < 0 , product of the roots > 0 and $D \geq 0$

i.e. $-\frac{b}{a} < 0, \frac{c}{a} > 0, b^2 - 4ac \geq 0$

- Both roots of $f(x) = 0$ are positive,
if sum of the roots > 0 , product of the roots > 0 and $D \geq 0$

i.e. $-\frac{b}{a} > 0, \frac{c}{a} > 0, b^2 - 4ac \geq 0$

- Roots of $f(x) = 0$ are opposite in sign,

if product of the roots < 0 i.e. $\frac{c}{a} < 0$

Illustration - 48 If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then :

(A) $a < 2$

(B) $2 \leq a \leq 3$

(C) $3 < a \leq 4$

(D) $a > 4$

SOLUTION : (A)

$$\text{Let } f(x) = x^2 - 2ax + a^2 + a - 3$$

As both roots of $f(x) = 0$ are less than 3, we can take $af(3) > 0$, $-b/2a < 3$ and $D \geq 0$ [Using section 6.4]

Consider $af(3) > 0$:

$$\Rightarrow 1[9 - 6a + a^2 + a - 3] > 0 \Rightarrow a^2 - 5a + 6 > 0 \Rightarrow a \in (-\infty, 2) \cup (3, \infty) \dots\dots\dots \text{(i)}$$

$$\text{Consider } -b/2a < 3: \frac{-(-2a)}{2} < 3 \Rightarrow a < 3 \dots\dots\dots \text{(ii)}$$

$$\text{Consider } D \geq 0: 4a - 4(a^2 + a - 3) \geq 0 \Rightarrow -4(a - 3) \geq 0 \Rightarrow a - 3 \leq 0$$

$$\Rightarrow a \in (-\infty, 3] \dots\dots\dots \text{(iii)}$$

Combining (i), (ii) and (iii) on the number line, we get: $a \in (-\infty, 2)$

Illustration - 49 The values of p for which the roots of the equation $(p - 3)x^2 - 2px + 5p = 0$ are real and positive are :

- (A) $p \in [2, 3]$ (B) $p \in \left(3, \frac{15}{4}\right]$ (C) $p \in (-\infty, 0) \cup (3, \infty)$ (D) $p \in \left(2, \frac{15}{4}\right)$

SOLUTION : (B)

The roots are real and positive if $D \geq 0$, sum of the roots > 0 and product of the roots > 0 . [Using result 6.7]

$D \geq 0$:

$$\Rightarrow 4p^2 - 20p(p - 3) \geq 0 \Rightarrow -4p^2 + 15p \geq 0 \Rightarrow 4p^2 - 15p \leq 0 \Rightarrow p \in [0, 15/4] \dots\dots\dots \text{(i)}$$

Sum of the roots > 0 :

$$\frac{2p}{p - 3} > 0 \Rightarrow \frac{p}{p - 3} > 0$$

$$\Rightarrow p(p - 3) > 0 \Rightarrow p \in (-\infty, 0) \cup (3, \infty) \dots\dots\dots \text{(ii)}$$

Product of the roots > 0 :

$$\frac{5p}{p - 3} > 0 \Rightarrow \frac{p}{p - 3} > 0 \Rightarrow p(p - 3) > 0 \Rightarrow p \in (-\infty, 0) \cup (3, \infty) \dots\dots\dots \text{(iii)}$$

Combining (i), (ii) and (iii) on the number line, we get: $p \in (3, 15/4]$.

Illustration - 50 The values of a for which $2x^2 - 2(2a + 1)x + a(a + 1) = 0$ may have one root less than a and other root greater than a are given by

- (A) $1 > a > 0$ (B) $-1 < a < 0$ (C) $a \geq 0$ (D) $a > 0$ or $a < -1$

SOLUTION : (D)

The given condition suggests that a lies between the roots. Let $f(x) = 2x^2 - 2(2a + 1)x + a(a + 1)$. For a to lie between the roots, we must have $f(a) < 0$

$$\Rightarrow 2a^2 - 2a(2a + 1) + a(a + 1) < 0$$

$$\Rightarrow -a^2 - a < 0 \Rightarrow a^2 + a > 0 \Rightarrow a > 0 \text{ or } a < -1$$

Illustration - 51 The values of a for which both the roots of $x^2 - 4ax + 2a^2 - 3a + 5 = 0$ are greater than 2, are

- (A) $a \in (1, \infty)$ (B) $a = 1$ (C) $a \in (-\infty, 1)$ (D) $a \in (9/2, \infty)$

SOLUTION : (D)

Let $f(x) = x^2 - 4ax + 2a^2 - 3a + 5$. The conditions for both the roots to exceed 2 are

(i) $D \geq 0$ (ii) $f(2) > 0$ and (iii) $x_v > 2$

Now consider $D \geq 0$

$$\Rightarrow 16a^2 - 4(2a^2 - 3a + 5) \geq 0 \Rightarrow 2a^2 + 3a - 5 \geq 0$$

$$\Rightarrow (2a+5)(a-1) \geq 0$$

$$\Rightarrow a \in (-\infty, -5/2] \cup [1, \infty) \quad \dots\dots(i)$$

Now consider $f(2) > 0$

$$\Rightarrow 4 - 8a + (2a^2 - 3a + 5) > 0 \Rightarrow 2a^2 - 11a + 9 > 0$$

$$\Rightarrow (2a-9)(a-1) > 0$$

$$\Rightarrow a \in (-\infty, 1) \cup \left(\frac{9}{2}, \infty\right) \quad \dots\dots(ii)$$

Now consider $x_v > 2$

$$\Rightarrow \frac{4a}{2} > 2$$

$$\Rightarrow a > 1 \quad \dots\dots(iii)$$

On combining (i), (ii) and (iii), we get : $a \in \left(\frac{9}{2}, \infty\right)$

TRANSFORMATION OF EQUATIONS

Section - 7

7.1 Transformation of an equation into another equation whose roots are the reciprocals of the roots of the given equation

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0 \quad \dots\dots(i)$

be the given equation. Let x and y be respectively the roots of the given equation and that of the transformed equation.

Then, $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

Putting $x = \frac{1}{y}$ in (i), we get :

$$\frac{a_0}{y^n} + \frac{a_1}{y^{n-1}} + \frac{a_2}{y^{n-2}} + \dots + \frac{a_{n-1}}{y} + a_n = 0 \Rightarrow a_n y^n + a_{n-1} y^{n-1} + \dots + a_1 y + a_0 = 0$$

This is the required equation.

Note : Thus, to obtain an equation whose roots are reciprocals of the roots of a given equation is obtained by replacing x by $1/x$ in the given equation.

Illustration - 52 Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ be in H.P.

(A) $27r^2 + 9pqr + 2q^3 = 0$

(B) $27r^2 - 9pqr + 2q^3 = 0$

(C) $27r^2 + 9pqr + q^3 = 0$

(D) $27r^2 - 9pqr + q^3 = 0$

SOLUTION : (B)

The equation whose roots are reciprocals of the roots of the given equation is given by

$$\frac{1}{x^3} - \frac{p}{x^2} + \frac{q}{x} - r = 0 \quad \text{or} \quad rx^3 - qx^2 + px - 1 = 0 \quad \dots\dots (i)$$

Since the roots of the given equation are in H.P. so, the roots of this equation are in A.P. Let its roots be $a-d$, a and $a+d$.

Then, $(a-d) + a + (a+d) = -\left(-\frac{q}{r}\right) \Rightarrow 3a = \frac{q}{r} \Rightarrow a = \frac{q}{3r}$

Since a is a root of (i), so,

$$\begin{aligned} ra^3 - qa^2 + pa - 1 &= 0 & \Rightarrow r\left(\frac{q}{3r}\right)^3 - q\left(\frac{q}{3r}\right)^2 + p\left(\frac{q}{3r}\right) - 1 &= 0 \\ \Rightarrow \frac{q^3}{27r^2} - \frac{q^3}{9r^2} + \frac{pq}{3r} - 1 &= 0 & \Rightarrow q^3 - 3q^3 + 9pqr - 27r^2 &= 0 \Rightarrow 27r^2 - 9pqr + 2q^3 = 0 \end{aligned}$$

7.2 Transformation of an equation into another equation whose roots are negatives of the roots of the given equation.

Let the given equation be

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

Note : Let x be a root of the given equation and y be a root of the transformed equation. Then, $y = -x$ or $x = -y$. Thus, the transformed equation is obtained by putting $x = -y$ in $f(x) = 0$ and is therefore $f(-y) = 0$

or $a_0 y^n - a_1 y^{n-1} + a_2 y^{n-2} + \dots + (-1)^n a_n = 0$

Illustrating the Concepts :

The equation whose roots are negative of the roots of the equation : $x^3 - 5x^2 - 7x - 3 = 0$

$$(-x)^3 - 5(-x)^2 - 7(-x) - 3 = 0 \text{ or } -x^3 - 5x^2 + 7x - 3 = 0 \text{ or } x^3 + 5x^2 - 7x + 3 = 0.$$

7.3 Transformation of an equation into another equation whose roots are square of the roots of a given equation

Let x be a root of the given equation and y be that of the transformed equation. Then,

$$y = x^2 \Rightarrow x = \sqrt{y}.$$

Note : Thus, an equation whose roots are squares of the roots of a given equation is obtained by replacing x by $\sqrt{x}$ in the given equation.

Illustrating the Concepts :

Form an equation whose roots are squares of the roots of the equation : $x^3 - 6x^2 + 11x - 6 = 0$.

Replacing x by $\sqrt{x}$ in the given equation, we get :

$$\begin{aligned} (\sqrt{x})^3 - 6(\sqrt{x})^2 + 11\sqrt{x} - 6 &= 0 \Rightarrow x^{3/2} + 11\sqrt{x} = 6x + 6 \Rightarrow \sqrt{x}(x + 11) = 6(x + 1) \\ \Rightarrow x(x + 11)^2 &= 36(x + 1)^2 \Rightarrow x^3 - 14x^2 + 49x - 36 = 0 \end{aligned}$$

7.4 Transformation of an equation into another equation whose roots are cubes of the roots of the given equation.

Let x be a root of the given equation and y be that of the transformed equation. Then, $y = x^3 \Rightarrow x = y^{1/3}$

Note : Thus, an equation whose roots are cubes of the roots of a given equation is obtained by replacing x by $x^{1/3}$ in the given equation.

Illustrating the Concepts :

Form an equation whose roots are cubes of the roots of equation : $ax^3 + bx^2 + cx + d = 0$.

Replacing x by $x^{1/3}$ in the given equation, we get

$$\begin{aligned} a(x^{1/3})^3 + b(x^{1/3})^2 + c(x^{1/3}) + d &= 0 \Rightarrow ax + d = -(bx^{2/3} + cx^{1/3}) \\ \Rightarrow (ax + d)^3 &= -(bx^{2/3} + cx^{1/3})^3 \\ \Rightarrow a^3 x^3 + 3a^2 dx^2 + 3ad^2 x + d^3 &= -\{b^3 x^2 + c^3 x + 3bcx(bx^{2/3} + cx^{1/3})\} \\ \Rightarrow a^3 x^3 + 3a^2 dx^2 + 3ad^2 x + d^3 &= -\{b^3 x^2 + c^3 x - 3bcx(ax + d)\} \\ \Rightarrow a^3 x^3 + x^2(3a^2 d - 3abc + b^3) + x(3ad^2 - 3bcd + c^3) + d^3 &= 0 \end{aligned}$$

This is the required equation.

7.5 Relations between Roots and Coefficients

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are roots of the equation

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0, \text{ then}$$

$$f(x) = a_0 (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \dots (x - \alpha_n)$$

$$\therefore a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = a_0 (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$$

Comparing the coefficients of x^{n-1} on both sides, we get :

$$S_1 = \alpha_1 + \alpha_2 + \dots + \alpha_n = \sum \alpha_i = \frac{-a_1}{a_0} \quad \text{or} \quad S_1 = -\frac{\text{coeff. of } x^{n-1}}{\text{coeff. of } x^n}$$

Comparing the coefficients of x^{n-2} on both sides, we get :

$$S_2 = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \dots = \sum_{i \neq j} \alpha_i \alpha_j = (-1)^2 \frac{a_2}{a_0} \quad \text{or} \quad S_2 = \frac{(-1)^2 \text{coeff. of } x^{n-2}}{\text{coeff. of } x^n}$$

Comparing the coefficients of x^{n-3} on both sides, we get :

$$S_3 = \alpha_1 \alpha_2 \alpha_3 + \alpha_2 \alpha_3 \alpha_4 + \dots = \sum_{i \neq j \neq k} \alpha_i \alpha_j \alpha_k = (-1)^3 \frac{a_3}{a_0}$$

or,

$$S_3 = \frac{(-1)^3 \text{coeff. of } x^{n-3}}{\text{coeff. of } x^n}$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$S_n = \alpha_1 \alpha_2 \alpha_3 \dots \alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{constant term}}{\text{coeff. of } x^n}$$

Here, S_k denotes the sum of the products of the roots taken k at a time.

Particular Cases :

Quadratic Equation : If α, β are roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Cubic Equation : If α, β, γ are roots of a cubic equation

$$ax^3 + bx^2 + cx + d = 0, \quad \text{then} \quad \alpha + \beta + \gamma = -b/a, \quad \alpha\beta + \beta\gamma + \gamma\alpha = (-1)^2 \frac{c}{a} = \frac{c}{a}$$

and

$$\alpha\beta\gamma = (-1)^3 \frac{d}{a} = -\frac{d}{a}.$$

Biquadratic Equation : If $\alpha, \beta, \gamma, \delta$ are roots of the biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$S_1 = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$

$$S_2 = \alpha\beta + \beta\gamma + \alpha\delta + \alpha\gamma + \beta\delta + \gamma\delta = (-1)^2 \frac{c}{a} = \frac{c}{a}$$

or,

$$S_2 = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a}$$

$$S_3 = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

or,

$$S_3 = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -\frac{d}{a} \quad \text{and,} \quad S_4 = \alpha\beta\gamma\delta = (-1)^4 \frac{e}{a} = \frac{e}{a}.$$

Illustration - 53 If the sum of two roots of the equation $x^3 - px^2 + qx - r = 0$ is zero, then:

- (A) $pq = r$ (B) $pr = q$ (C) $qr = p$ (D) None of these

SOLUTION : (A)

Let the roots of the given equation be α, β, γ such that $\alpha + \beta = 0$. Then,

$$\alpha + \beta + \gamma = -\frac{(-p)}{1} \Rightarrow \alpha + \beta + \gamma = p \Rightarrow \gamma = p \quad [\because \alpha + \beta = 0]$$

But γ is a root of the given equation. Therefore,

$$\gamma^3 - p\gamma^2 + q\gamma - r = 0 \Rightarrow p^3 - p^3 + qp - r = 0 \Rightarrow pq = r$$

Illustration - 54 Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in A.P.

(A) $2p^3 + 9pq + 27r = 0$

(B) $p^3 + 9pq + 27r = 0$

(C) $2p^3 - 9pq + 27r = 0$

(D) $p^3 - 9pq + 27r = 0$

SOLUTION : (A)

Let the roots of the given equation be $a - d, a, a + d$.

Then,

$$(a - d) + a + (a + d) = \frac{-(-p)}{1} \Rightarrow a = p/3$$

Since a is a root of the given equation. Therefore,

$$a^3 - pa^2 + qa - r = 0 \Rightarrow \frac{p^3}{27} - \frac{p^3}{9} + \frac{qp}{3} - r = 0 \Rightarrow 2p^3 - 9pq + 27r = 0$$

This is the required condition.